Approximations of Weyl fractional-order integrals with insurance applications*

Chengxiu Ling[a], Zuoxiang Peng[b]

[a]Department of Actuarial Science, University of Lausanne,

UNIL-Dorigny, 1015 Lausanne, Switzerland

[b]School of Mathematics and Statistics, Southwest University, 400715 Chongqing, China

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Abstract: In this paper, we investigate the approximations of generalized Weyl fractional-order integrals in extreme value theory framework. We present three applications of our asymptotic results concerning the higher-order tail approximations of deflated risks as well as approximations of Haezendonck-Goovaerts and expectile risk measures. Illustration of the obtained results is done by various examples and some numerical analysis.

Key words and phrases: Weyl fractional integrals; deflated risks; expectile; Haezendonck-Goovaerts risk measure; second-order/third-order regularly variations

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1 Introduction

Throughout this paper, let X be a random risk with distribution function (df) F (denoted by $X \sim F$), and $S \in (0,1)$ an independent of random risk. Of interest is the following integral $\mathcal{I}_{\mathcal{L},S}(x,X)$ given by

$$\mathcal{I}_{\mathcal{L},S}(x,X) := \mathbb{E}\left\{\mathcal{L}(X)\mathbb{I}\{SX > x\}\right\}$$

for some given function $\mathcal{L}(\cdot)$ such that the integral is well-defined. Here $\mathbb{I}\{\cdot\}$ stands for the indicator function. The integral $\mathcal{I}_{\mathcal{L},S}(x,X)$ is closely related to the Weyl fractional-order integral; see e.g., [21, 29, 31] for related applications on beta random scaling and Wicksell problem.

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In various theoretical and practical situations, the question arising naturally is how the approximation of $\mathcal{I}_{\mathcal{L},S}(x,X)$ as x goes to the right-endpoint of F, is influenced by the tail behavior of S and X. For instance [19, 20] studied the asymptotics of the tail of deflated risk SX which is reduced by $\mathcal{I}_{\mathcal{L},S}(x,X)$ with $\mathcal{L}(x)\equiv 1$.
Another motivation for considering the approximations of $\mathcal{I}_{\mathcal{L},S}(x,X)$ comes from finance and risk management fields. Particularly, if S is a Beta distributed random variable with parameters 1 and $\kappa, \kappa>0$ (denoted by $X \sim Beta(1,\kappa)$), and $\mathcal{L}(x)=x^{\kappa}$, then, with $x_+ = \max(x,0)$

$$\mathcal{I}_{\mathcal{L},S}(x,X) = \mathbb{E}\left\{(X-x)_+^{\kappa}\right\},$$

which is closely related to several risk measures such as the Haezendonck-Goovaerts (H-G) and expectile risk measures; see [2, 4, 5, 24, 25, 33], and references therein for related discussions.

In this paper, we are interested in the derivation of some approximations of the integral $\mathcal{I}_{\mathcal{L},S}(x,X)$ with $\mathcal{L}(x)=x^{\kappa}$ for some given constant κ, abbreviate it as $\mathcal{I}_{\kappa,S}(x,X)$, i.e.,

$$\mathcal{I}_{\kappa,S}(x,X) := \mathbb{E}\left\{X^{\kappa}\mathbb{I}\{SX>x\}\right\}. \tag{1.1}$$

We remark that one could similarly consider the general function $\mathcal{L}(\cdot)$ by applying the methodology of regular variations; for some technical reasons we study only the power function.

Our principle results, Theorem 2.1 and Theorem 2.3, are concerned with the second- and third-order approximations of (1.1). Our main methodology is based on the higher-order regular variation theory which was discussed deeply by [12, 15, 36]. As a by-product, we establish a technical inequality, namely the extensional Drees' inequality for third-order regular variations (see Lemma 6.1 in the Appendix), which is of its own interest; see e.g., [10, 13, 15, 23, 26, 36] for related discussions.
As important applications of these results, we shall discuss the tail asymptotics of deflated risks in Theorem 3.1, and Theorem 3.3 (which deal with the cases that X has Weibull and Gumbel tails in a unified way), refining those in [19, 20] where they only studied the first- and second-order tail approximations of SX under three different tail behavior of S and X. Moreover, with the aid of Corollary 2.5, dealing with the higher-order approximations of (1.1) for $S \sim Beta(1,\kappa)$, we investigate the approximations of H-G and expectile risk measures which were initially studied by [33] and [2], respectively. The two competitive risk measures have received more and more attention due to their nice mathematical and statistics properties; see e.g., [5, 9, 34]. As expected, our Theorem 3.5 and Theorem 3.7 are significant refinements of findings displayed in [24, 33] and [2, 25].

The rest of this paper is organized as follows. In Section 2, we establish our main results following by the three applications including the approximations of deflated risks as well as higher-order approximations of H-G and expectile risk measures. Section 4 is devoted to several illustrated examples and some numerical analysis. All the proofs are

relegated to Section 5. We conclude this paper with an Appendix containing a technical inequality.

2 Main Results

We start with the definitions and some properties of regular variations which are key to establish our main results.

A measurable function $f : [0, \infty) \to \mathbb{R}$ is said to be an extended regularly varying function (ERV) at infinity with index $\gamma \in \mathbb{R}$, denoted by $f \in \mathrm{ERV}_\gamma$, if (cf. [6, 10])

$$\lim_{t\to\infty} \frac{f(tx) - f(t)}{a(t)} = \frac{x^\gamma - 1}{\gamma} := D_\gamma(x) \tag{2.1}$$

holds for all $x > 0$ and an eventually positive function $a(\cdot)$, which is referred to as the auxiliary function. In the meanwhile, f is regularly varying with index γ, denoted by $f \in \mathrm{RV}_\gamma$, if the limit in (2.1) holds with $(f(tx)-f(t))/a(t)$ and $D_\gamma(x)$ replaced by $f(tx)/f(t)$ and x^γ, respectively.

ERV and RV are powerful tools in the study of extreme value of statistics since it provides a suitable framework to study key features and properties of dfs belonging to max-domains of attractions. Namely, a df F is said to be in the max-domain attraction (MDA) of $G_\gamma(x) := \exp\left(-(1+\gamma x)_+^{-1/\gamma}\right), \gamma \in \mathbb{R}$, i.e., there exist some constants $a_n > 0, b_n \in \mathbb{R}$ such that

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} |F^n(a_n x + b_n) - G_\gamma(x)| = 0,$$

which holds if and only if $U \in \mathrm{ERV}_\gamma$ with $U(t) := F^{\leftarrow}(1 - 1/t)$ the tail quantile function; see [10, 14]. The df F is so-called in the Fréchet, Gumbel and Weibull MDA according to $\gamma > 0, \gamma = 0$ and $\gamma < 0$, respectively.

In this paper, we mainly use the second- and third-order ERV and RV extensions, which are mainly used to investigate the speed of convergence of the first- and second-order expansions of certain quantities of interest in different contexts; see e.g., [8, 11, 22, 24, 28, 30].

Refining (2.1), we say that f is of second-order extended regular variation with parameters $\gamma \in \mathbb{R}$ and $\rho \le 0$, denoted by $f \in 2\mathrm{ERV}_{\gamma,\rho}$, if there exist some auxiliary functions $a(\cdot)$ eventually positive, and $A(\cdot)$ with constant sign near infinity satisfying $\lim_{t\to\infty} A(t) = 0$, such that for all $x > 0$ (cf. [12, 32])

$$\lim_{t\to\infty} \frac{(f(tx) - f(t))/a(t) - D_\gamma(x)}{A(t)} = \int_1^x y^{\gamma-1} \int_1^y u^{\rho-1}\, du\, dy := H_{\gamma,\rho}(x). \tag{2.2}$$

Further, we shall write $f \in 3\mathrm{ERV}_{\gamma,\rho,\eta}$ meaning that f is of third-order regular variation with parameters $\gamma \in \mathbb{R}$ and $\rho, \eta \le 0$, if there exist first-, second- and third-order auxiliary functions $a(\cdot)$ eventually positive, and $A(\cdot), B(\cdot)$ with constant sign near infinity satisfying $\lim_{t\to\infty} A(t) = \lim_{t\to\infty} B(t) = 0$, such that for all $x > 0$ (cf. [15, 36])

$$\lim_{t\to\infty} \frac{\frac{(f(tx)-f(t))/a(t) - D_\gamma(x)}{A(t)} - H_{\gamma,\rho}(x)}{B(t)} = \int_1^x y^{\gamma-1} \int_1^y u^{\rho-1} \int_1^u v^{\eta-1}\, dv\, du\, dy := R_{\gamma,\rho,\eta}(x). \tag{2.3}$$

Similarly, we define $f \in 2\mathrm{RV}$ (or $f \in 3\mathrm{RV}$) with auxiliary function $A(\cdot)$ (and $B(\cdot)$) if the limit in (2.2) (or (2.3)) holds with $(f(tx) - f(t))/a(t)$ and $D_\gamma(x), H_{\gamma,\rho}(x)$ (and $R_{\gamma,\rho,\eta}(x)$) replaced by $f(tx)/f(t)$ and $x^\gamma, x^\gamma D_\rho(x)$ (and $x^\gamma D_{\rho+\eta}(x)$), respectively.

Note in passing that for $f \in 3\mathrm{RV}_{\gamma,\rho,\eta}$ with auxiliary functions $A(\cdot)$ and $B(\cdot)$, we have from [15] that $A \in 2\mathrm{RV}_{\rho,\eta}$ with auxiliary function B, and $|B| \in \mathrm{RV}_\eta$.

Throughout this paper, we write $\overline{Q} := 1 - Q$ for some function Q and $U(t) := F^{\leftarrow}(1-1/t), t \geq 1$ for the tail quantile function of X. By $\Gamma(\cdot)$ and $B(\cdot,\cdot)$ we mean the Euler Gamma function and Beta function. All the limits are taken as the argument goes to $x_F = U(\infty)$, the right endpoint of X unless otherwise stated.

Our first result, Theorem 2.1, investigates the approximations of $\mathcal{I}_{\kappa,S}(x,X)$ given by (1.1) for X being in the Fréchet MDA. Further, for $\kappa < \alpha, \varrho, \varsigma \leq 0$, we denote $d_{0,\kappa} = \mathbb{E}\left\{S^{\alpha-\kappa}\right\}$ and

$$d_{1,\kappa} = \frac{\mathbb{E}\left\{S^{\alpha-\kappa-\varrho}\right\} - \mathbb{E}\left\{S^{\alpha-\kappa}\right\}}{\varrho} + \frac{\kappa\mathbb{E}\left\{S^{\alpha-\kappa-\varrho}\right\}}{\alpha(\alpha-\kappa-\varrho)}, \quad d_{2,\kappa} = \frac{\kappa(\mathbb{E}\left\{S^{\alpha-\kappa-2\varrho}\right\} - \mathbb{E}\left\{S^{\alpha-\kappa-\varrho}\right\})}{\alpha\varrho(\alpha-\kappa-\varrho)}$$
$$d_{3,\kappa} = \frac{\kappa}{\alpha}\left(\frac{\mathbb{E}\left\{S^{\alpha-\kappa-\varrho-\varsigma}\right\} - \mathbb{E}\left\{S^{\alpha-\kappa-\varrho}\right\}}{(\alpha-\kappa-\varrho)\varsigma} + \frac{\mathbb{E}\left\{S^{\alpha-\kappa-\varrho-\varsigma}\right\}}{\alpha-\kappa-\varrho-\varsigma}\right) + \frac{\mathbb{E}\left\{S^{\alpha-\kappa-\varrho-\varsigma}\right\} - \mathbb{E}\left\{S^{\alpha-\kappa}\right\}}{\varrho+\varsigma}.$$

Theorem 2.1. *(i) If $\overline{F} \in 2\mathrm{RV}_{-\alpha,\varrho}$ with auxiliary function A for some $\alpha > 0$ and $\varrho \leq 0$, then, for $\mathcal{I}_{\kappa,S}(x,X)$ given by* (1.1) *with $\kappa < \alpha$*

$$\frac{\mathcal{I}_{\kappa,S}(x,X)}{x^\kappa \overline{F}(x)} = \frac{\alpha}{\alpha-\kappa}\left(d_{0,\kappa} + d_{1,\kappa}A(x)(1+o(1))\right). \tag{2.4}$$

(ii) If $\overline{F} \in 3\mathrm{RV}_{-\alpha,\varrho,\varsigma}$ with auxiliary functions A and B, then

$$\frac{\mathcal{I}_{\kappa,S}(x,X)}{x^\kappa \overline{F}(x)} = \frac{\alpha}{\alpha-\kappa}\left(d_{0,\kappa} + A(x)\big(d_{1,\kappa} + d_{2,\kappa}A(x)(1+o(1)) + d_{3,\kappa}B(x)(1+o(1))\big)\right). \tag{2.5}$$

Remark 2.2. *(i) Recalling that the Weyl fractional-order integral $\mathcal{J}_{\beta+1,K_c}(x,X)$ with weight function $K_c(x) := x^c$ is given by (cf. [21])*

$$\mathcal{J}_{\beta+1,K_c}(x,X) = \frac{\mathcal{I}_{\kappa,S}(x,X)}{\Gamma(\beta+1)}, \quad \text{with } S \sim Beta(1,\beta),\ \kappa = \beta + c,$$

an immediate application of (2.4) *and* (2.5) *together with $\mathbb{E}\left\{S^l\right\} = lB(l+1,\beta+1),\ l > 0$ implies the second- and third-order expansions of $\mathcal{J}_{\beta+1,K_c}(x,X)$ extending Theorem 7.2 in [21].*

(ii) We see that the speed of convergence of the second-order expansion is determined by the two auxiliary functions A and B, i.e., the parameters ρ and ϱ. Most common risks are in the third-order Hall class defined by (3.1) *below, i.e., satisfy the third-order regularly varying conditions with equal ϱ and ς; see [7, 8].*

Next, we consider that X is in the Gumbel and Weibull MDA. Hereafter, denote below for $\gamma \in \mathbb{R}, \alpha > 0, \rho, \eta \leq 0, \varrho < 0$

with $c_{\alpha,l}=\alpha(\alpha-1)\cdots(\alpha-l+1)/l!, l\in\mathbb{N}$ and $D_\gamma, H_{\gamma,\rho}, R_{\gamma,\rho,\eta}$ given by (2.3)

$$\begin{cases} L_\alpha=\int_0^1(D_\gamma(1/s))^\alpha\,ds, \quad M_{\alpha,l}=c_{\alpha,l}\int_0^1(D_\gamma(1/s))^{\alpha-l}(H_{\gamma,\rho}(1/s))^l\,ds \\ N_{\alpha,l,\varrho}=c_{\alpha,l}\int_0^1(D_\gamma(1/s))^{\alpha-l}(H_{\gamma,\rho}(1/s))^l\frac{(D_\gamma(1/s))^{-\varrho}-1}{\varrho}\,ds \\ Q_\alpha=\alpha\int_0^1(D_\gamma(1/s))^{\alpha-1}R_{\gamma,\rho,\eta}(1/s)\,ds. \end{cases} \tag{2.6}$$

Theorem 2.3. *(i) If $U\in 2\mathrm{ERV}_{\gamma,\rho}$ with auxiliary functions a,A for some $\gamma,\rho\le 0$, and $\overline{G}(1-1/x)\in 2\mathrm{RV}_{-\alpha,\varrho},\alpha>0,\varrho<0$ with auxiliary function $\widetilde{A}$, then*

$$\frac{\mathcal{I}_{\kappa,S}(x,X)}{x^\kappa\overline{F}(x)\overline{G}\left(1-1/\varphi_t\right)}=L_\alpha+\left(M_{\alpha,1}A(t)+N_{\alpha,0,\varrho}\widetilde{A}(\varphi_t)+(\kappa-\alpha)\frac{L_{\alpha+1}}{\varphi_t}\right)(1+o(1)). \tag{2.7}$$

(ii) If $U\in 3\mathrm{ERV}_{\gamma,\rho,\eta}$ with auxiliary functions a,A and B for some $\gamma,\rho,\eta\le 0$, and $\overline{G}(1-1/x)\in 3\mathrm{RV}_{-\alpha,\varrho,\varsigma},\alpha>0,\varrho,\varsigma<0$ with auxiliary functions $\widetilde{A}$ and $\widetilde{B}$, then

$$\begin{aligned}\frac{\mathcal{I}_{\kappa,S}(x,X)}{x^\kappa\overline{F}(x)\overline{G}\left(1-1/\varphi_t\right)}&=L_\alpha+M_{\alpha,1}A(t)+N_{\alpha,0,\varrho}\widetilde{A}(\varphi_t)+(\kappa-\alpha)\frac{L_{\alpha+1}}{\varphi_t}\\ &+\Bigg(A(t)\Big(M_{\alpha,2}A(t)+Q_\alpha B(t)+N_{\alpha,1,\varrho}\widetilde{A}(\varphi_t)+(\kappa-\alpha)\frac{M_{\alpha+1,1}}{\varphi_t}\Big)+(\kappa-\alpha)(\kappa-\alpha-1)\frac{L_{\alpha+2}}{2\varphi_t^2}\\ &+\widetilde{A}(\varphi_t)\left(N_{\alpha,0,\varrho+\varsigma}\widetilde{B}(\varphi_t)+\left((\kappa-\alpha)N_{\alpha+1,0,\varrho}+L_{\alpha-\varrho+1}\right)\frac{1}{\varphi_t}\right)\Bigg)(1+o(1))\end{aligned} \tag{2.8}$$

with $\varphi_t=U(t)/a(t), t=1/\overline{F}(x)$ and $L_\alpha, M_{\alpha,l}, N_{\alpha,l,\varrho}, Q_\alpha$ given by (2.6).

Remark 2.4. *(i) It is possible to allow ϱ,ς to be non-positive for $\gamma<0$.*

(ii) We see that Theorem 2.3 conducts a unified way in terms of the tail quantile function. Further, the speed of convergence seems more involved in the related parameters and auxiliary functions.

Next, we specify the expansions of $\mathcal{I}_{\kappa,S}(x,X)$ where we consider $S\sim Beta(1,\kappa)$ and certain third-order regularly varying conditions are imposed on X (which is helpful to calculate explicitly the coefficients involved). In the meanwhile, we will see that Corollary 2.5 given below is of crucial importance in the derivation of the tail asymptotics of H-G and expectile risk measures; see Sections 3.2 and 3.3.

In what follows, set for $\gamma\ne 0,\rho\le 0$

$$\begin{aligned}\xi_{\kappa,\rho}&=B\left(\frac{1-\rho}{\gamma}-\kappa,\kappa\right)\mathbb{I}\{\gamma>0\}+B\left(1-\frac{1-\rho}{\gamma},\kappa\right)\mathbb{I}\{\gamma<0\}\\ \widetilde{M}_{\kappa,1}&=\frac{1}{\gamma\rho}\left(\frac{\xi_{\kappa,\rho}}{\xi_{\kappa,0}}-\frac{\xi_{\kappa-1,\rho}}{\xi_{\kappa-1,0}}\right),\quad \Delta_\kappa=\frac{1}{\gamma\rho}\left(\kappa\frac{\xi_{\kappa-1,\rho}}{\xi_{\kappa-1,0}}-(\kappa-1)\frac{\xi_{\kappa,\rho}}{\xi_{\kappa,0}}-1\right).\end{aligned} \tag{2.9}$$

Corollary 2.5. *If (i) $U\in 3\mathrm{RV}_{\gamma,\rho,\eta}$ with auxiliary functions A and B with $\gamma>0,\rho,\eta\le 0$, or (ii) $x_F-U\in 3\mathrm{RV}_{\gamma,\rho,\eta}$ with $\gamma<0,\rho,\eta\le 0$ and auxiliary functions A and B, then for all $\kappa>0$ such that $\kappa\gamma<1$*

$$\frac{\mathbb{E}\left\{(X-U(t))_+^\kappa\right\}}{t^{-1}(a(t))^\kappa}=L_\kappa+M_{\kappa,1}A(t)+M_{\kappa,2}A^2(t)(1+o(1))+Q_\kappa A(t)B(t)(1+o(1)),\quad t\to\infty \tag{2.10}$$

holds with $a(t)=\gamma U(t)\mathbb{I}\{\gamma>0\}-\gamma(x_F-U(t))\mathbb{I}\{\gamma<0\}, L_\kappa=\kappa\xi_{\kappa,0}/|\gamma|^\kappa$ and

$$M_{\kappa,1}=\frac{\kappa\mathrm{sign}(\gamma)}{|\gamma|^{\kappa+1}\rho}\left(\xi_{\kappa,\rho}-\xi_{\kappa,0}\right),\ Q_\kappa=\frac{\kappa\mathrm{sign}(\gamma)}{|\gamma|^{\kappa+1}(\rho+\eta)}\left(\xi_{\kappa,\rho+\eta}-\xi_{\kappa,0}\right).$$

$$M_{\kappa,2}=\frac{\kappa}{2|\gamma|^{\kappa+2}\rho^2}\big((1-2\rho-\gamma)\xi_{\kappa,2\rho}-2(1-\rho-\gamma)\xi_{\kappa,\rho}+(1-\gamma)\xi_{\kappa,0}\big).$$

Remark 2.6. *Note that the coefficients in* (2.10) *are understood as their limits when ρ or η are zeros. Specifically, we have, with $l=1,2$*

$$\lim_{\rho\to 0}M_{\kappa,l}=\frac{c_{\kappa,l}\mathrm{sign}(\gamma)}{|\gamma|^{\kappa+l}}\omega_{\kappa,l},\quad \lim_{\eta\to 0}Q_\kappa=\frac{\kappa\mathrm{sign}(\gamma)}{|\gamma|^{\kappa+1}}\left(\omega_{\kappa,1}\mathbb{I}\{\rho\neq 0\}+\frac{\widetilde{\omega}_\kappa}{2|\gamma|}\mathbb{I}\{\rho=0\}\right),$$

where

$$\omega_{\kappa,l}=\frac{1}{|\gamma|}\int_0^\infty x^l(1-e^{-x})^{\kappa-l}\exp\left(-\frac{1-\kappa\gamma\mathbb{I}\{\gamma>0\}-\gamma\mathbb{I}\{\gamma<0\}}{|\gamma|}x\right)dx$$
$$\widetilde{\omega}_\kappa=\frac{1}{|\gamma|}\int_0^\infty x^2(1-e^{-x})^{\kappa-1}\exp\left(-\frac{1-\kappa\gamma\mathbb{I}\{\gamma>0\}-\gamma\mathbb{I}\{\gamma<0\}}{|\gamma|}x\right)dx.$$

3 Applications

In this section, we present three applications in insurance fields, namely the higher-order tail expansions of the deflated risks which refine those in [19], and approximations of H-G and expectile risk measures.

3.1 Asymptotic expansions of deflated risks

In the following, we apply Theorem 2.1 and Theorem 2.3 with $\kappa=0$ to obtain the third-order expansions of the tail of deflated risk SX refining those in [19, 20].

Theorem 3.1. *Under the conditions as in Theorem 2.1 (ii), we have*

$$\frac{\mathbb{P}\{SX>x\}}{\overline{F}(x)}=\mathbb{E}\{S^\alpha\}+\frac{\mathbb{E}\{S^{\alpha-\varrho}\}-\mathbb{E}\{S^\alpha\}}{\varrho}A(x)+\frac{\mathbb{E}\{S^{\alpha-\varrho-\varsigma}\}-\mathbb{E}\{S^\alpha\}}{\varrho+\varsigma}A(x)B(x)(1+o(1)).$$

Example 3.2. *(Third-order Hall-class)* Let X be a random variable with a df F such that, for some $\alpha,b>0,\varrho<0$ and $c,d\neq 0$

$$\overline{F}(x)=bx^{-\alpha}(1+cx^\varrho+dx^{2\varrho}(1+o(1))),\quad x\to\infty, \tag{3.1}$$

i.e., F is in the third-order Hall-class; see e.g., [7, 8, 22]. It follows then by Proposition 6.3 that $\overline{F}\in 3\mathrm{RV}_{-\alpha,\varrho,\varrho}$ with auxiliary functions A and B given by

$$A(x)=\frac{\varrho cx^\varrho}{1+cx^\rho},\quad B(x)=\frac{2d}{c}x^\varrho.$$

An immediate application of Theorem 3.1 with an independent scaling factor $S\in(0,1)$ yields that

$$\overline{H}(x):=\mathbb{P}\{SX>x\}=\mathbb{E}\{S^\alpha\}bx^{-\alpha}\left(1+\frac{\mathbb{E}\{S^{\alpha-\varrho}\}}{\mathbb{E}\{S^\alpha\}}cx^\varrho+\frac{\mathbb{E}\{S^{\alpha-2\varrho}\}}{\mathbb{E}\{S^\alpha\}}dx^{2\varrho}(1+o(1))\right),\quad x\to\infty,$$

which together with Proposition 6.3 yields that the Value-at-Risk of SX at level q, denoted by $\mathrm{VaR}_q(SX)(:= \overline{H}^{\leftarrow}(1-q))$, equals with $c_q = \Big(\frac{b\mathbb{E}\{S^\alpha\}}{1-q}\Big)^{1/\alpha}$

$$\frac{\mathrm{VaR}_q(SX)}{c_q} = 1 + \frac{c\mathbb{E}\left\{S^{\alpha-\varrho}\right\}}{\alpha\mathbb{E}\left\{S^\alpha\right\}}c_q^\varrho + \left(\frac{1}{2}\Big(\frac{c\mathbb{E}\left\{S^{\alpha-\varrho}\right\}}{\alpha\mathbb{E}\left\{S^\alpha\right\}}\Big)^2(1-\alpha+2\varrho) + \frac{d\mathbb{E}\left\{S^{\alpha-2\varrho}\right\}}{\alpha\mathbb{E}\left\{S^\alpha\right\}}\right)c_q^{2\varrho}(1+o(1)), \quad q\uparrow 1.$$

Theorem 3.3. *Under the conditions as in Theorem 2.3 (ii), we have*

$$\begin{aligned}\frac{\mathbb{P}\left\{SX > x\right\}}{\overline{F}(x)\overline{G}\left(1-1/\varphi_t\right)} &= L_\alpha + M_{\alpha,1}A(t) + N_{\alpha,0,\varrho}\widetilde{A}(\varphi_t) - \frac{\alpha L_{\alpha+1}}{\varphi_t}\\ &+A(t)\Big(M_{\alpha,2}A(t) + Q_\alpha B(t) + N_{\alpha,1,\varrho}\widetilde{A}(\varphi_t)\Big) + N_{\alpha,0,\varrho+\varsigma}\widetilde{A}(\varphi_t)\widetilde{B}(\varphi_t)(1+o(1))\\ &-\frac{\alpha}{\varphi_t}\left(M_{\alpha+1,1}A(t) - (\alpha+1)\frac{L_{\alpha+2}}{2\varphi_t} + \Big(\frac{N_{\alpha+1,0,\varrho}}{\alpha} - L_{\alpha-\varrho+1}\Big)\widetilde{A}(\varphi_t)\right)(1+o(1))\end{aligned}$$

with $\varphi_t = U(t)/a(t), t = 1/\overline{F}(x)$ and $L_\alpha, M_{\alpha,l}, N_{\alpha,l,\varrho}, Q_\alpha$ given by (2.6).

Remark 3.4. *We see that Theorem 3.3 refines the second-order asymptotic expansions of deflated risks in [19] in a unified form, see Theorems 2.3 and 2.6 therein.*

3.2 Asymptotic expansions of Haezendonck-Goovaerts risk measure

Haezendonck-Goovaerts (H-G) risk measure is based on premium calculation principle via Orlicz norm which was first introduced by [18]. It is shown by [3, 4] that H-G risk measure is a law-invariant and coherent risk measure and thus an challenging alternative to Value-at-Risk and Expected Shortfall.

In what follows, with the aid of Corollary 2.5 we shall establish the higher-order expansions of H-G risk measure which refine those by [24, 33]. It is shown (see Proposition 1.1 in [24] or [33]) that, for a Young function $\phi(t) = t^\kappa, \kappa\ge 1$, and X a risk variable with $\mathbb{E}\left\{X_+^\kappa\right\} < \infty$ and $\mathbb{P}\left\{X = x_F\right\} = 0$, then the H-G risk measure for X at level $q\in(0,1)$, denoted by $H_q[X]$, is given by

$$H_q[X] = x + \Big(\frac{\mathbb{E}\left\{(X-x)_+^\kappa\right\}}{1-q}\Big)^{1/\kappa}, \tag{3.2}$$

where $x = x(q)\in(-\infty, x_F)$ is the unique solution to the equation

$$\frac{(\mathbb{E}\left\{(X-x)_+^{\kappa-1}\right\})^\kappa}{(\mathbb{E}\left\{(X-x)_+^\kappa\right\})^{\kappa-1}} = 1-q, \quad \kappa > 1 \tag{3.3}$$

and $x = \mathrm{VaR}_q(X)(:= F^{\leftarrow}(q))$ for $\kappa = 1$.

For simplicity of notation, denote, with $\xi_{\kappa,\rho}, \widetilde{M}_{\kappa,1}, \Delta_\kappa$ given in (2.9)

$$\begin{cases} \mathrm{c} = \kappa\Big(\frac{1-\kappa\gamma}{\kappa|\gamma|}\Big)^\kappa\xi_{\kappa,0}; \quad c_0 = \frac{\mathrm{c}^\gamma}{1-\kappa\gamma}; \quad c_1 = \frac{1}{\rho}\left(\mathrm{c}^\rho\frac{\xi_{\kappa,\rho}}{\xi_{\kappa,0}} - 1\right)\\ c_2 = \gamma\mathrm{c}^{2\rho}\left(\frac{1}{2\gamma^2\rho^2}\left[(1-\gamma-2\rho)\frac{\xi_{\kappa,2\rho}}{\xi_{\kappa,0}} - 2(1-\gamma-\rho)\frac{\xi_{\kappa,\rho}}{\xi_{\kappa,0}} + 1-\gamma\right]\right.\\ \qquad +\Delta_\kappa\left[\kappa(\gamma+\rho)\widetilde{M}_{\kappa,1} + \left(\rho+\frac{\gamma-1}{2}\right)\Delta_\kappa + \frac{1}{\gamma}\right]\\ \qquad \left.+\kappa\widetilde{M}_{\kappa,1}\left[\frac{\kappa-1}{2}\widetilde{M}_{\kappa,1} - \frac{1}{\gamma\rho}\left(\frac{\xi_{\kappa-1,\rho}}{\xi_{\kappa-1,0}} - 1\right)\right]\right) + \mathrm{c}^\rho\frac{\mathrm{c}^\rho-1}{\rho^2}\left(\frac{\xi_{\kappa,\rho}}{\xi_{\kappa,0}} - 1\right)\\ c_3 = \frac{1}{\rho+\eta}\left(\mathrm{c}^{\rho+\eta}\frac{\xi_{\kappa,\rho+\eta}}{\xi_{\kappa,0}} - 1\right) + \frac{\mathrm{c}^\rho(\mathrm{c}^\eta-1)}{\rho\eta}\left(\frac{\xi_{\kappa,\rho}}{\xi_{\kappa,0}} - 1\right). \end{cases} \tag{3.4}$$

Theorem 3.5. *Let $H_q[X]$ be defined by (3.2). We have (i) If $U \in 3\mathrm{RV}_{\gamma,\rho,\eta}, 0 < \kappa\gamma < 1$ and $\rho, \eta \le 0$ with auxiliary functions A and B, then*

$$H_q[X] = c_0 F^{\leftarrow}(q)\left(1 + c_1\epsilon_q + c_2\epsilon_q^2(1+o(1)) + c_3\epsilon_q\psi_q(1+o(1))\right), \quad q \uparrow 1.$$

(ii) If $x_F - U \in 3\mathrm{RV}_{\gamma,\rho,\eta}, \gamma < 0$ and $\rho, \eta \le 0$ with auxiliary functions A and B, then

$$x_F - H_q[X] = c_0(x_F - F^{\leftarrow}(q))\left(1 + c_1\epsilon_q + c_2\epsilon_q^2(1+o(1)) + c_3\epsilon_q\psi_q(1+o(1))\right), \quad q \uparrow 1,$$

where $\mathrm{c}, c_i, 0 \le i \le 3$ are given as in (3.4) and $\epsilon_q := A(1/(1-q)), \psi_q := B(1/(1-q))$.

Remark 3.6. *(i) Clearly, our Theorem 3.5 provides the third-order asymptotics of $H_q[X]$ which is based on the bias-adjustment of the second-order ones; see Example 4.1 and 4.2.*

(ii) Theorem 3.5 refines those in [24].

3.3 Asymptotic expansions of expectile risk measure

Expectiles are first introduced in [27] in a statistical context. Namely, for a random variable X with finite expectation the expectile $e_q = e_q[X], q \in [0,1]$ is defined as the unique minimizer of an asymmetric quadratic loss function as follows

$$e_q = \arg\min_{x\in\mathbb{R}} \left(q\mathbb{E}\left\{(X-x)_+^2\right\} + (1-q)\mathbb{E}\left\{(x-X)_+^2\right\}\right)$$

or equivalently as the unique solution of the first-order condition

$$e_q = \mathbb{E}\{X\} + \frac{2q-1}{1-q}\mathbb{E}\{(X-e_q)_+\}, \tag{3.5}$$

see, e.g., [2, 5] for further discussions. Several generalizations of expectiles and its application are extensively studied in both statistical and actuarial literature; see, e.g., [17, 35, 37].

In the following, we investigate the higher-order expansions of e_q extending those by [2, 25]. For simplicity of notation, we denote for $0 < \gamma < 1, \rho, \eta \le 0$ and $D = (\gamma^{-1}-1)^{-\rho}/(1-\gamma-\rho)$

$$\begin{aligned}
d_0 &= \gamma(\gamma^{-1}-1)^{\gamma}\mathbb{E}\{X\}, \quad d_1 = -2\gamma, \quad d_2 = D + \frac{(\gamma^{-1}-1)^{-\rho}-1}{\rho},\\
d_3 &= 2(\gamma^2-\gamma), \quad d_4 = D^2\left(\frac{\gamma-1}{2\gamma} + \frac{\rho}{\gamma}\right) + D\left(\left(\frac{1}{\rho}+\frac{1}{\gamma}\right)\left(\frac{\gamma}{1-\gamma}\right)^{\rho} - \frac{1}{\rho}\right),\\
d_5 &= -2(1+\rho)D - 2\frac{(\gamma^{-1}-1)^{-\rho}-1}{\gamma^{-1}\rho} - 2(\gamma^{-1}-1)^{-\rho},\\
d_6 &= D\left(\frac{1-\gamma-\rho)(\gamma^{-1}-1)^{-\eta}}{1-\gamma-\rho-\eta} + \frac{(\gamma^{-1}-1)^{-\eta}-1}{\eta}\right) + \frac{(\gamma^{-1}-1)^{-\rho-\eta}-1}{\rho+\eta},\\
d_7 &= \frac{(\gamma^{-1}-1)^{2\gamma+1}}{2}(\gamma\mathbb{E}\{X\})^2, \quad d_8 = -2\gamma^2(\gamma^{-1}-1)^{\gamma+1}\mathbb{E}\{X\},\\
d_9 &= \frac{2\gamma(\gamma^{-1}-1)^{\gamma+1-\rho}}{1-\gamma-\rho}\mathbb{E}\{X\}.
\end{aligned}$$

Theorem 3.7. *If $U \in 2\mathrm{RV}_{\gamma,\rho}, \gamma \in (0,1), \rho \le 0$ with auxiliary function A, then as $q \uparrow 1$*

$$e_q = \Big(\frac{\gamma}{1-\gamma}\Big)^{\gamma} F^{\leftarrow}(q)\left(1 + \left(\frac{d_0}{F^{\leftarrow}(q)} + d_1(1-q) + d_2\epsilon_q\right)(1+o(1))\right).$$

If further $U \in 3\mathrm{RV}_{\gamma,\rho,\eta},\ \gamma \in (0,1), \rho, \eta \le 0$ with auxiliary functions A and B. Then, as $q \uparrow 1$

$$\begin{aligned} e_q =& \Big(\frac{\gamma}{1-\gamma}\Big)^{\gamma} F^{\leftarrow}(q)\Bigg(1 + \frac{d_0}{F^{\leftarrow}(q)} + d_1(1-q) + d_2\epsilon_q \\ &+ d_3(1-q)^2(1+o(1)) + \epsilon_q(d_4\epsilon_q + d_5(1-q) + d_6\psi_q)(1+o(1)) \\ &+ \frac{1}{F^{\leftarrow}(q)}\left(\frac{d_7}{F^{\leftarrow}(q)} + d_8(1-q) + d_9\epsilon_q\right)(1+o(1))\Bigg), \end{aligned}$$

with $\epsilon_q := A(1/(1-q)), \psi_q := B(1/(1-q))$

Remark 3.8. *(i) The speed of the second-order expansion is determined by $\max(\rho - 1, 2\rho, \rho + \eta)$ for $\mathbb{E}\{X\} = 0$, see Example 4.2; otherwise by $\max(-2\gamma, 2\rho, \rho + \eta)$ and thus we may neglect the terms related to d_3, d_5 and d_8, see Example 4.1.*

(ii) Theorem 3.7 extends Theorem ? in [25]. Moreover, it is worth mentioning that their results for $\mathbb{E}\{X\} \neq 0$ are only available for $\rho \ge -1$ since the location transformation will change the second-order parameter; see [7].

Next, we consider the case that $X \sim F$ which is in the Weibull MDA.

Theorem 3.9. *If $x_F - U \in 2\mathrm{RV}_{\gamma,\rho}, \gamma, \rho < 0$ with auxiliary function A such that, with a constant $C > 0$*

$$x_F - U(t) = Ct^{\gamma}\left(1 + \frac{A(t)}{\rho}(1+o(1))\right),$$

then, with $\alpha = -1/\gamma$, $x_0 = (\alpha+1)(x_F - \mathbb{E}\{X\})$

$$\begin{aligned} x_F - e_q \quad = \quad & (C^{\alpha}x_0(1-q))^{1/(\alpha+1)}\Bigg(1 - \frac{(C^{\alpha}x_0(1-q))^{1/(\alpha+1)}}{\alpha(x_F - \mathbb{E}\{X\})}(1+o(1)) \\ &+ \frac{(\alpha+1)(C/x_0)^{\alpha\rho/(\alpha+1)}}{\rho(\alpha+1-\alpha\rho)}A\left((1-q)^{-\frac{\alpha}{\alpha+1}}\right)(1+o(1))\Bigg). \end{aligned}$$

Remark 3.10. *(i) Theorem 3.9 generalizes Proposition 2.5 in [2]. Moreover, one may refine the second-order results above by imposing $3RV$ conditions on $x_F - U$..*

(ii) Numerous examples of F that satisfy the conditions of Theorem 3.3 and Theorem 3.9 are presented in [19].

(iii) One can follow the similar arguments for H-G and expectile risk measures above to consider the case that X has a Weibull-type tails.

4 Examples and Numerical Analysis

Example 4.1. *(Burr distribution)* Let X be a Burr distributed random variable with parameters $a, b > 0$, i.e., $\overline{F}(x) = (1 + x^a)^{-b}, x \ge 0$. It follows that (cf. [1])

$$\overline{F}(x) = x^{-ab}\left(1 - bx^{-a} + \frac{b(b+1)}{2}x^{-2a}(1+o(1))\right), \quad x \to \infty.$$

Consequently, it follows by Proposition 6.3 (see also Table 1 in [7]) with $\alpha = -ab, \rho = -a$ that

$$U(t) = \overline{F}^{\leftarrow}(1/t) = t^{1/(ab)}\left(1 - \frac{t^{-1/b}}{a} + \frac{1-a}{2a^2}t^{-2/b}(1+o(1))\right).$$

Consequently, $U \in 3\mathrm{RV}_{1/(ab),-1/b,-1/b}$ with auxiliary functions A, B given by

$$A(t) = \frac{t^{-1/b}}{ab - bt^{-1/b}}, \quad B(t) = \frac{a-1}{a}t^{-1/b}.$$

For expectiles, note that $\mathbb{E}\{X\} = a^{-1}B(b - 1/a, 1/a)$ for $ab > 1$. By Theorem 3.7 we have the first-, second- and third-order approximations of e_q as $q \uparrow 1$. In Figure 1 we take $a = 2, b = 1.5$ and we see the higher-order approximation of e_q is more accurate than the lower-order ones.

For H-G risk measure, we take $a = 1/2, b = 4$ and $\kappa = 1.5$, and thus $\gamma = 1/2, \rho = \eta = -1/4$. By Theorem 3.5

$$\begin{aligned}
\frac{H_q[X]}{F^{\leftarrow}(q)} &= 2.6935\left(1 + 1.9312\cdot\frac{(1-q)^{1/4}}{2(1-2(1-q)^{1/4})} + \frac{(1-q)^{1/4}}{2}\left(-0.1822\cdot\frac{(1-q)^{1/4}}{2} - 1.2575(1-q)^{1/4}\right)(1+o(1))\right)\\
&= 2.6935\left(1 + 0.5116(1-q) + 0.2587(1-q)^2(1+o(1))\right), \quad q \uparrow 1.
\end{aligned}$$

Therefore, we see that the first-, and second-order approximations slightly underestimate the $H_q[X]$ while the third-order approximation has smaller error.

Example 4.2. *(Student t-distribution)* Let $X \sim t_v$ be a student t-distributed random variable with $v > 1$ degrees of freedom. We have $\mathbb{E}\{X\} = 0$ and its probability density function f is given by

$$\begin{aligned}
f(x) &= \frac{\Gamma((v+1)/2)}{\sqrt{v\pi}\Gamma(v/2)}\left(1 + \frac{x^2}{v}\right)^{-(v+1)/2}\\
&= \frac{\Gamma((v+1)/2)}{\sqrt{v\pi}\Gamma(v/2)}v^{(v+1)/2}x^{-(v+1)}\left(1 - \frac{v(v+1)}{2}x^{-2} + \frac{v^2(v+1)(v+3)}{8}x^{-4}(1+o(1))\right), \quad x \to \infty\\
&=: C_v x^{-k_1}(1 + k_2 x^{\rho'} + k_3 x^{2\rho'}(1+o(1))) \quad \text{with } C_v := v^{v/2}/B(v/2, 1/2).
\end{aligned}$$

It follows from Lemma 6.1 and further by the dominated convergence theorem that

$$\begin{aligned}
\overline{F}(x) &= xf(x)\int_1^\infty \frac{f(tx)}{f(x)}\,dt\\
&= xf(x)\int_1^\infty t^{-k_1}\left(1 + \frac{k_2 x^{\rho'}(t^{\rho'}-1)}{1 + k_2 x^{\rho'} + k_3 x^{2\rho'}(1+o(1))} + k_3 x^{2\rho'}(t^{2\rho'}-1)(1+o(1))\right)dt\\
&= \frac{xf(x)}{k_1 - 1}\left(1 + \frac{\rho'}{k_1 - 1 - \rho'}\frac{k_2 x^{\rho'}}{1 + k_2 x^{\rho'} + k_3 x^{2\rho'}} + \frac{2\rho'}{k_1 - 1 - 2\rho'}k_3 x^{2\rho'}(1+o(1))\right)\\
&= \frac{C_v}{v}x^{-v}\left(1 - \frac{v^2(v+1)}{2(v+2)}x^{-2} + \frac{v^3(v+1)(v+3)}{8(v+4)}x^{-4}(1+o(1))\right), \quad x \to \infty.
\end{aligned}$$

Therefore, it follows from Proposition 6.3 that (see also Table 1 in [7])

$$U(t) = \overline{F}^{\leftarrow}(1/t) = \left(\frac{C_v t}{v}\right)^{1/v}\left(1 - \frac{v(v+1)}{2(v+2)}\left(\frac{C_v t}{v}\right)^{-2/v} - \frac{v^3(v+1)(v+3)}{8(v+2)^2(v+4)}\left(\frac{C_v t}{v}\right)^{-4/v}(1+o(1))\right).$$

Consequently, we have $U \in 3\mathrm{RV}_{1/v,-2/v,-2/v}$ with auxiliary functions A and B given by

$$A(t) = \frac{(v+1)(C_v t/v)^{-2/v}}{v + 2 - v(v+1)(C_v t/v)^{-2/v}/2}, \quad B(t) = \frac{v^2(v+3)}{2(v+2)(v+4)}\left(\frac{C_v t}{v}\right)^{-2/v}.$$

We have by Theorem 3.5 the second-order approximation with a convergent rate $\max(-1,-2/v)$. We obtain further by Theorem 3.7 the third-order approximations of e_q as $q\uparrow 1$ with the speed of convergence of the second-order approximation as $\max(-2/v-1,-4/v)$. In Figure 2, we take $v=1.2$ and we see that the second- and third-order approximations of e_q is much better than the first-order approximations. Particularly, when $q=0.9979$, the third-, second- and first-order evaluations are $(261.0483, 261.0483, 261.9426)$ for the true value $e_q=261.0483$.

For Haezendonck-Goovaerts risk measure, taking $v=2,\kappa=1.1$, and thus $C_2=1$. We have by Theorem 3.5

$$
\begin{aligned}
\frac{H_q[X]}{F^{\leftarrow}(q)} &= 2.1044\left(1+0.6822\cdot\frac{3(1-q)}{4-3(1-q)}+\frac{3(1-q)}{4}\left(-0.125\cdot\frac{3(1-q)}{4}+0.3998\cdot\frac{5}{6}(1-q)\right)(1+o(1))\right)\\
&= 2.1044\left(1+0.5116(1-q)+0.5634(1-q)^2(1+o(1))\right),\quad q\uparrow 1.
\end{aligned}
$$

Therefore, as in Example 4.1 we see that the first-, and second-order approximations slightly underestimate the $H_q[X]$ while the third-order approximation has smaller error.

Example 4.3. *(Beta distribution)* Let $X\sim Beta(a,b), a,b>0$ with probability density function given by

$$
f(x)=\frac{1}{B(a,b)}x^{a-1}(1-x)^{b-1},\quad 0<x<1,\ a,\ b>0.
$$

Note that

$$
\overline{F}\left(1-\frac{1}{t}\right)=\int_{1-1/t}^{1}f(x)\,dx=\int_t^\infty\frac{1}{s^2}f\left(1-\frac{1}{s}\right)\,ds,
$$

with

$$
\frac{1}{t^2}f\left(1-\frac{1}{t}\right)=\frac{t^{-(b+1)}}{B(a,b)}\left(1-\frac{a-1}{t}+\frac{(a-1)(a-2)}{2t^2}(1+o(1))\right),\quad t\to\infty.
$$

Therefore, by the dominated convergence theorem

$$
\overline{F}\left(1-\frac{1}{t}\right)=\frac{t^{-b}}{bB(a,b)}\left(1-\frac{b(a-1)}{b+1}\frac{1}{t}+\frac{b(a-1)(a-2)}{2(b+2)}\frac{1}{t^2}(1+o(1))\right)=:g(t).
$$

It follows by Proposition 6.3 and $1/(1-U(t))=g^{\leftarrow}(1/t)$ that

$$
\begin{aligned}
1-U(t) &= \Big(\frac{t}{bB(a,b)}\Big)^{-1/b}\left(1+\frac{a-1}{b+1}\Big(\frac{t}{bB(a,b)}\Big)^{-1/b}\right.\\
&\quad\left.+\frac{a-1}{b+1}\left(\frac{a-1}{b+1}+\frac{a+b}{2(b+2)}\right)\Big(\frac{t}{bB(a,b)}\Big)^{-2/b}(1+o(1))\right),\quad t\to\infty.
\end{aligned}
$$

Consequently, we have $1-U\in 3\mathrm{RV}_{-1/b,-1/b,-1/b}$ with auxiliary functions A and B given by

$$
\begin{aligned}
A(t) &= -\frac{a-1}{b(b+1)}\Big(\frac{t}{bB(a,b)}\Big)^{-1/b}\Big/\left[1+\frac{a-1}{b+1}\Big(\frac{t}{bB(a,b)}\Big)^{-1/b}\right]\\
B(t) &= 2\left(\frac{a-1}{b+1}+\frac{a+b}{2(b+2)}\right)\Big(\frac{t}{bB(a,b)}\Big)^{-1/b}.
\end{aligned}
$$

For H-G risk measure, we take $a=3,b=6$ and $\kappa=2$, thus $\gamma=\rho=\eta=-1/6$. By Theorem 3.5

$$
\frac{1-H_q[X]}{1-F^{\leftarrow}(q)} = 0.8055\left(1-0.9254\cdot\frac{1}{21}\frac{((1-q)/28)^{1/6}}{1-2/7((1-q)/28)^{1/6}}\right)
$$

$$
\begin{aligned}
& -\frac{1}{21}\Big(\frac{1-q}{28}\Big)^{1/6}\bigg(\frac{1.98931}{21}\Big(\frac{1-q}{28}\Big)^{1/6}+1.1547\cdot\frac{95}{56}\Big(\frac{1-q}{28}\Big)^{1/6}\bigg)(1+o(1))\bigg) \\
= \;& 0.8055\left(1-0.0253(1-q)^{1/6}-0.0364(1-q)^{1/3}(1+o(1))\right), \quad q\uparrow 1.
\end{aligned}
$$

Therefore, we see that the first-, and second-order approximations slightly overestimate the $1-H_q[X]$ while the third-order approximation has smaller error. In particular, for $a=b=1$, we have $C=\alpha=1,\rho=-\infty$

$$
1-e_q=\sqrt{1-q}\big(1-2\sqrt{1-q}(1+o(1))\big), \quad q\uparrow 1
$$

which coincides with the true value of $e_q=(q-\sqrt{q-q^2})/(2q-1)$ (see Example 3.1 in [2]). In Figure 3, we take $(a,b)=(2,3)$ and we see that our second-order approximation of $e_q[X]$ performs very well.

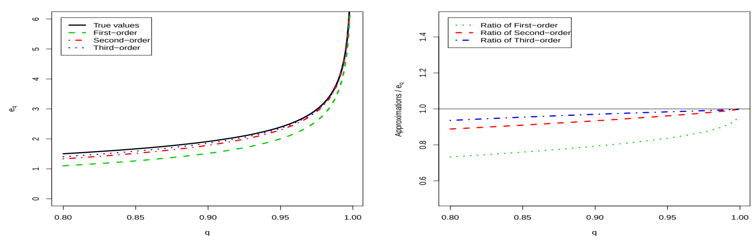

Figure 1: Approximations of expectile $e_q[X], q\uparrow 1$ (left) and Ratios of approximations divided by the true values of $e_q[X]$ (right) for $X\sim Burr(a,b)$ with $(a,b)=(2,1.5)$.

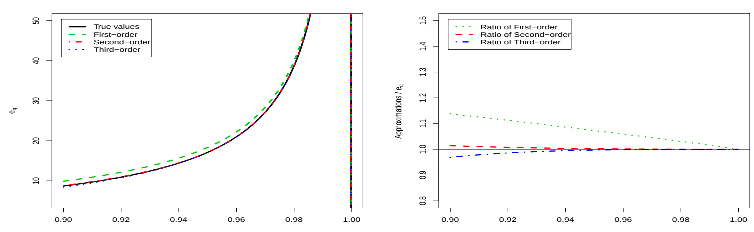

Figure 2: Approximations of expectile $e_q[X], q\uparrow 1$ (left) and Ratios of approximations divided by the true values of $e_q[X]$ (right) for $X\sim t_v$ with $v=1.2$.

5 Proofs

Proof of Theorem 2.1 Noting that S and X are independent, we have

$$
\begin{aligned}
\frac{\mathbb{E}\left\{X^{\kappa}\mathbb{I}\{SX>x\}\right\}}{\overline{F}(x)} &= \frac{1}{\overline{F}(x)}\int_0^1 \mathbb{E}\left\{X^{\kappa}\mathbb{I}\{sX>x\}\right\} dG(s) \\
&= \frac{1}{\overline{F}(x)}\int_0^1\int_{x/s}^{\infty} y^{\kappa}\, dF(y)\, dG(s)
\end{aligned}
$$

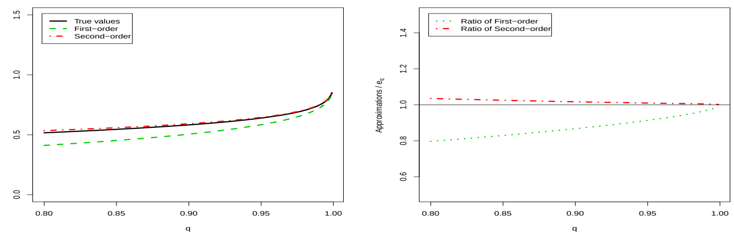

Figure 3: Approximations of expectile $e_q[X], q \uparrow 1$ (left) and Ratios of approximations divided by the true values of $e_q[X]$ (right) with $X \sim Beta(2,3)$.

$$= \quad x^\kappa \int_0^1 s^{-\kappa} \frac{\overline{F}(x/s)}{\overline{F}(x)} \left(1 + \kappa \int_1^\infty y^{\kappa-1} \frac{\overline{F}((x/s)y)}{\overline{F}(x/s)}\, dy\right) dG(s).$$

It follows from Lemma 5.2 in [13], for any given $\varepsilon > 0$, there exists $x_0 > 0$ such that for all $x > x_0$ and all $s \in (0,1)$ and $y > 1$

$$\begin{cases} \left|\frac{\overline{F}(x/s)}{\overline{F}(x)} - s^\alpha \left(1 + \frac{s^{-\varrho}-1}{\varrho} A(x)\right)\right| \le \varepsilon |A(x)|(1 + s^\alpha + 2s^{\alpha-\varrho-\varepsilon}) \\ \left|\frac{\overline{F}((x/s)y)}{\overline{F}(x/s)} - y^{-\alpha} \left(1 + \frac{y^{\varrho}-1}{\varrho} A(x/s)\right)\right| \le \varepsilon |A(x/s)|(1 + y^{-\alpha} + 2y^{-\alpha+\varrho+\varepsilon}). \end{cases} \tag{5.1}$$

Further a straightforward application of Potter bounds for $A(\cdot)$ (cf. Proposition B.1.9 in [10]) yields that $A(x/s)/A(x) \le (1+\varepsilon)s^{-\varrho-\varepsilon}$. Consequently, by the dominated convergence theorem

$$\begin{aligned} \frac{\mathbb{E}\left\{X^\kappa \mathbb{I}\{SX > x\}\right\}}{x^\kappa \overline{F}(x)} &= \int_0^1 s^{\alpha-\kappa} \left(1 + \frac{s^{-\varrho}-1}{\varrho} A(x)(1+o(1))\right) \\ &\times \left(1 + \kappa \int_1^\infty y^{\kappa-\alpha-1} \left(1 + \frac{y^\varrho - 1}{\varrho} s^{-\varrho} A(x)(1+o(1))\right) dy\right) dG(s) \end{aligned}$$

where the $o(1)$-terms are uniform for all $s \in (0,1)$ and $y \in (1,\infty)$. We complete the proof of (2.4).

To prove (2.5), we use by Lemma 6.1

$$\begin{aligned} &\left|\frac{\overline{F}(x/s)}{\overline{F}(x)} - s^\alpha \left(1 + \frac{s^{-\varrho}-1}{\varrho} A(x) + \frac{s^{-\varrho-\varsigma}-1}{\varrho+\varsigma} A(x)B(x)\right)\right| \\ &\qquad \le \varepsilon |A(x)||B(x)|(1 + s^\alpha + 2s^{\alpha-\varrho} + 4s^{\alpha-\varrho-\varsigma-\varepsilon}) \end{aligned}$$

$$\begin{aligned} &\left|\frac{\overline{F}((x/s)y)}{\overline{F}(x/s)} - y^{-\alpha} \left(1 + \frac{y^{\varrho}-1}{\varrho} A(x/s) + \frac{y^{\varrho+\varsigma}-1}{\varrho+\varsigma} A(x/s)B(x/s)\right)\right| \\ &\qquad \le \varepsilon |A(x)||B(x)|(1 + y^{-\alpha} + 2y^{-\alpha+\varrho} + 4y^{-\alpha+\varrho+\varsigma+\varepsilon}) \end{aligned}$$

and

$$\begin{aligned} \left|\frac{A(x/s)}{A(x)} - s^{-\varrho}\left(1 + \frac{s^{-\varsigma}-1}{\varsigma} B(x)\right)\right| &\le \varepsilon |B(x)|(1 + s^{-\varrho} + 2s^{-\varrho-\varsigma-\varepsilon}) \\ \frac{A(x/s)B(x/s)}{A(x)B(x)} &\le (1+\varepsilon)s^{-\varrho-\varsigma-\varepsilon}. \end{aligned}$$

Thus again using the dominated convergence theorem

$$\frac{\mathbb{E}\left\{X^\kappa \mathbb{I}\{SX > x\}\right\}}{x^\kappa \overline{F}(x)} = \int_0^1 s^{\alpha-\kappa} \left(1 + \frac{s^{-\varrho}-1}{\varrho} A(x) + \frac{s^{-\varrho-\varsigma}-1}{\varrho+\varsigma} A(x)B(x)(1+o(1))\right)$$

$$
\begin{aligned}
&\times \left(1+\kappa \int_1^\infty y^{\kappa-\alpha-1}\left(1+\frac{y^\varrho-1}{\varrho}A\Big(\frac{x}{s}\Big)+\frac{y^{\varrho+\varsigma}-1}{\varrho+\varsigma}A\Big(\frac{x}{s}\Big)B\Big(\frac{x}{s}\Big)(1+o(1))\right)\,dy\right)\,dG(s)\\
= \quad & \frac{\alpha}{\alpha-\kappa}\int_0^1 s^{\alpha-\kappa}\left(1+\frac{s^{-\varrho}-1}{\varrho}A(x)+\frac{s^{-\varrho-\varsigma}-1}{\varrho+\varsigma}A(x)B(x)(1+o(1))\right)\\
&\times\left(1+\frac{\kappa s^{-\varrho}}{\alpha(\alpha-\kappa-\varrho)}A(x)+\frac{\kappa}{\alpha}\left(\frac{s^{-\varrho}(s^{-\varsigma}-1)}{(\alpha-\kappa-\varrho)\varsigma}+\frac{s^{-\varrho-\varsigma}}{\alpha-\kappa-\varrho-\varsigma}\right)A(x)B(x)(1+o(1))\right)\,dG(s)\\
=: \quad & \frac{\alpha}{\alpha-\kappa}\left(d_{0,\kappa}+d_{1,\kappa}A(x)+d_{2,\kappa}A^2(x)(1+o(1))+d_{3,\kappa}A(x)B(x)(1+o(1))\right)
\end{aligned}
$$

establishing our proof with elementary consideration. □

Proof of Theorem 2.3 Letting $t=1/\overline{F}(x)$ and thus $x=U(t)$ for large t, we have

$$
\begin{aligned}
\frac{\mathbb{E}\{X^\kappa\mathbb{I}\{SX>x\}\}}{x^\kappa\overline{F}(x)} &= \int_0^1\Big(\frac{U(t/s)}{U(t)}\Big)^\kappa\mathbb{P}\left\{S>1-\frac{U(t/s)-U(t)}{U(t/s)}\right\}ds\\
&= \int_0^1\Big(\frac{U(t/s)}{U(t)}\Big)^\kappa\mathbb{P}\left\{S>1-\frac{U(t/s)-U(t)}{a(t)}\frac{a(t)}{U(t/s)}\right\}ds\\
&= \overline{G}\left(1-\frac{1}{\varphi_t}\right)\int_0^1\left(1+\frac{q_t(s)}{\varphi_t}\right)^\kappa\frac{\overline{G}\left(1-\frac{q_t(s)}{q_t(s)+\varphi_t}\right)}{\overline{G}\left(1-\frac{1}{\varphi_t}\right)}ds\\
&= \overline{G}\left(1-\frac{1}{\varphi_t}\right)\int_0^1(q_t(s))^\alpha\left(1+\frac{q_t(s)}{\varphi_t}\right)^{\kappa-\alpha}\frac{L\left(1+\frac{\varphi_t}{q_t(s)}\right)}{L(\varphi_t)}ds \qquad (5.2)
\end{aligned}
$$

with

$$
q_t(s)=\frac{U(t/s)-U(t)}{a(t)},\quad \varphi_t=\frac{U(t)}{a(t)},\quad L(z)=z^\alpha\overline{G}\left(1-\frac{1}{z}\right).
$$

Further

$$
\begin{aligned}
&\frac{\mathbb{E}\{X^\kappa\mathbb{I}\{SX>x\}\}}{x^\kappa\overline{F}(x)\overline{G}(1-1/\varphi_t)} = \int_0^1(D_\gamma(1/s))^\alpha\,ds+\int_0^1\left((q_t(s))^\alpha-(D_\gamma(1/s))^\alpha\right)ds\\
&\quad +\int_0^1(q_t(s))^\alpha\left(\left(1+\frac{q_t(s)}{\varphi_t}\right)^{\kappa-\alpha}-1\right)ds+\int_0^1(q_t(s))^\alpha\left(1+\frac{q_t(s)}{\varphi_t}\right)^{\kappa-\alpha}\left(\frac{L\left(1+\frac{\varphi_t}{q_t(s)}\right)}{L(\varphi_t)}-1\right)ds\\
&=: L_\alpha+\int_0^1(I_{1t}(s)+I_{2t}(s)+I_{3t}(s))\,ds. \qquad (5.3)
\end{aligned}
$$

Next, we will present the proofs of (2.7) and (2.8) by dealing with $I_{it}(s), i=1,2,3$ one by one.

(i) Proof of (2.7). It follows from Lemma 5.2 in [13] that, for any given $\varepsilon\in(0,1)$, there exists $t_0=t_0(\varepsilon)>0$ such that for all $t>t_0$ and all $s\in(0,1)$ we have $|q_t(s)-D_\gamma(1/s)|\le\varepsilon(1+s^{-\gamma}+2s^{-\gamma-\varepsilon})$. Furthermore, by Taylor's expansion $(1+x)^\alpha=1+\alpha x(1+o(1))$ for smaller $|x|$

$$
\begin{aligned}
\Big|\frac{I_{1t}(s)}{A(t)}\Big| &= (D_\gamma(1/s))^\alpha\Big|\frac{\left(1+\frac{q_t(s)-D_\gamma(1/s)}{D_\gamma(1/s)}\right)^\alpha-1}{A(t)}\Big|\\
&\le \alpha(D_\gamma(1/s))^{\alpha-1}\Big|\frac{q_t(s)-D_\gamma(1/s)}{A(t)}\Big|\left(1+\varepsilon(1+s^{-\gamma}+2s^{-\gamma-\varepsilon})\right)\\
&\le \alpha(D_\gamma(1/s))^{\alpha-1}\left(H_{\gamma,\rho}(1/s)+\varepsilon(1+s^{-\gamma}+2s^{-\gamma-\rho-\varepsilon})\right)\left(1+\varepsilon(1+s^{-\gamma}+2s^{-\gamma-\varepsilon})\right).
\end{aligned}
$$

which is integrable in $(0,1)$. Thus by the dominated convergence theorem

$$\int_0^1 I_{1t}(s)\,ds = A(t)\alpha\int_0^1 (D_\gamma(1/s))^{\alpha-1}H_{\gamma,\varrho}(1/s)\,ds(1+o(1)) =: M_{\alpha,1}A(t)(1+o(1)), \quad t\to\infty. \tag{5.4}$$

For $I_{2t}(s)$, noting that $\varphi_t\to\infty$ as $t\to\infty$ and using for small $x\geq 0$ that $|(1+x)^l-1|\leq 2|l|x$

$$\varphi_t|I_{2t}(s)|\leq 2|\kappa-\alpha|(q_t(s))^{\alpha+1}. \tag{5.5}$$

Thus, using again the dominated convergence theorem, we have

$$\int_0^1 I_{2t}(s)\,ds = \frac{1}{\varphi_t}(\kappa-\alpha)L_{\alpha+1}(1+o(1)), \quad t\to\infty. \tag{5.6}$$

It remains to deal with the third term $I_{3t}(s)$. By (5.5), for large t and all $s\in(0,1)$

$$\Big|\frac{I_{3t}(s)}{\widetilde{A}(\varphi_t)}\Big|\leq\frac{(q_t(s))^\alpha}{|\widetilde{A}(\varphi_t)|}\Big|\frac{L\left(1+\frac{\varphi_t}{q_t(s)}\right)}{L\left(\varphi_t\right)}-1\Big|\left(1+2|\kappa-\alpha|\frac{q_t(s)}{\varphi_t}\right).$$

We consider the two cases a) $\gamma<0$ and b) $\gamma=0$ separately.

a) For $\gamma<0$, recalling that $1/\varphi_t+1/q_t(s)>-\gamma$, using again Lemma 5.2 in [13] for $L\in 2\mathrm{RV}_{0,\varrho}$ that, for any given $\varepsilon>0$, there exists $t_0=t_0(\varepsilon)>0$ such that for all $\varphi(t)>t_0$ and all $s\in(0,1)$

$$\begin{aligned}\Big|\frac{I_{3t}(s)}{\widetilde{A}(\varphi_t)}\Big| &\leq (q_t(s))^\alpha\left(1+2|\kappa-\alpha|\frac{q_t(s)}{\varphi_t}\right)\\ &\times\left(\frac{(1/\varphi_t+1/q_t(s))^\varrho-1}{\varrho}+\varepsilon\left(1+\left(\frac{1}{\varphi_t}+\frac{1}{q_t(s)}\right)^\varrho\exp\left(\varepsilon\Big|\ln\left(\frac{1}{\varphi_t}+\frac{1}{q_t(s)}\right)\Big|\right)\right)\right)\end{aligned}$$

which is integrable.

b) For $\gamma=0$, it follows from Lemma 5.2 in [13] that $|I_{3t}(s)/\widetilde{A}(\varphi_t)|$ is integrable over $\mathbb{S}_t=\{s\in(0,1):\min(\varphi_t,1+\varphi_t/q_t(s))>t_0\}$. And on $\mathbb{S}_t^c=\{s\in(0,1):\varphi_t>t_0>1+\varphi_t/q_t(s)\}$, we use the similar arguments for proving Theorem 2.3 in [19] (see (5.6) therein) as follows.

Clearly $L(1+\varphi_t/q_t(s))\leq(1+\varphi_t/q_t(s))^\alpha\leq t_0^\alpha$. Moreover, by Theorem 1.10 in [16] that $\lim_{t\to\infty}(L(t)-L(\infty))/\widetilde{A}(t)=1/\varrho$ where $L(\infty):=\lim_{t\to\infty}L(t)$ and $\varrho<0$ implying $L(\infty)>0$. Therefore $L(\varphi_t)>L(\infty)/2$ for large t. Meanwhile, by Potter bounds, for any given $\varepsilon\in(0,1)$ and $\varphi_t>t_0>1+\varphi_t/q_t(s)$

$$|\widetilde{A}(\varphi_t)|\geq|\widetilde{A}((t_0-1)q_t(s))|\geq(1-\varepsilon)(q_t(s))^{\varrho-\varepsilon}|\widetilde{A}(t_0-1)|. \tag{5.7}$$

Consequently

$$\Big|\frac{I_{3t}(s)}{\widetilde{A}(\varphi_t)}\Big|\leq(q_t(s))^{\alpha-\varrho+\varepsilon}\left(1+2|\kappa-\alpha|\frac{q_t(s)}{t_0}\right)\frac{2t_0^\alpha/L(\infty)+1}{(1-\varepsilon)|\widetilde{A}(t_0-1)|}$$

holds on $\mathbb{S}_t^c$ whose Lebesgue measure goes to zero as $t\to\infty$.

Hence, a straightforward application of the dominated convergence theorem to $I_{3t}(s)$ for both a) $\gamma<0$ and b) $\gamma=0$ yields that

$$\frac{1}{\widetilde{A}(\varphi_t)}\int_0^1 I_{3t}(s)\,ds \quad=\quad \int_0^1(q_t(s))^\alpha\left(1+\frac{q_t(s)}{\varphi_t}\right)^{\kappa-\alpha}\frac{1}{\widetilde{A}(\varphi_t)}\left(\frac{L\left(1+\frac{\varphi_t}{q_t(s)}\right)}{L\left(\varphi_t\right)}-1\right)ds$$

$$
\begin{aligned}
&= \int_0^1 \left(D_\gamma\Big(\frac{1}{s}\Big)\right)^\alpha \frac{(D_\gamma(1/s))^{-\varrho}-1}{\varrho}\,ds(1+o(1)) \\
&=: N_{\alpha,0,\varrho}(1+o(1)), \quad t\to\infty
\end{aligned} \tag{5.8}
$$

which together with (5.3), (5.4) and (5.6) establishes the proof of (2.7).

(ii) Proof of (2.8). In the following, we use the same notation aforementioned as before. For $I_{1t}(s)$ given in (5.3), using $(1+x)^\alpha = 1+\alpha x + c_{\alpha,2}x^2(1+o(1))$ for smaller x and Lemma 6.1, there exists a constant $C>0$, for any $\varepsilon>0$ there exists some $t_0=t_0(\varepsilon)>1$ such that for all $t>t_0$ and $s\in(0,1)$

$$
\begin{aligned}
&\frac{\Big|I_{1t}(s)/A(t)-\alpha(D_\gamma(1/s))^{\alpha-1}H_{\gamma,\rho}(1/s)\Big|}{|A(t)|+|B(t)|} \\
&\le \alpha(D_\gamma(1/s))^{\alpha-1}\frac{\Big|\frac{q_t(s)-D_\gamma(1/s)}{A(t)}-H_{\gamma,\rho}(1/s)\Big|}{|B(t)|}\frac{|B(t)|}{|A(t)|+|B(t)|} \\
&\quad +c_{\alpha,2}(D_\gamma(1/s))^{\alpha-2}\Big(\frac{q_t(s)-D_\gamma(1/s)}{A(t)}\Big)^2\frac{|A(t)|}{|A(t)|+|B(t)|}\big(1+\varepsilon(1+s^{-\gamma}+2s^{-\gamma-\varepsilon})\big) \\
&\le \alpha(D_\gamma(1/s))^{\alpha-1}\big(R_{\gamma,\rho,\eta}(1/s)+\varepsilon(1+s^{-\gamma}+2s^{-\gamma-\rho}+4s^{-\gamma-\rho-\eta-\varepsilon}+s^{-C}\mathbb{I}\{\gamma=\rho=0\})\big) \\
&\quad +c_{\alpha,2}(D_\gamma(1/s))^{\alpha-2}\big(H_{\gamma,\rho}(1/s)+\varepsilon(1+s^{-\gamma}+2s^{-\gamma-\rho-\varepsilon})\big)^2\big(1+\varepsilon(1+s^{-\gamma}+2s^{-\gamma-\varepsilon})\big).
\end{aligned}
$$

Consequently, again by the dominated convergence theorem

$$
\int_0^1 I_{1t}(s)\,ds = A(t)\Big(M_{\alpha,1}+B(t)Q_\alpha(1+o(1))+A(t)M_{\alpha,2}(1+o(1))\Big), \quad t\to\infty. \tag{5.9}
$$

For $I_{2t}(s)$, note that $|(1+x)^p-1-px|\le Cx^2$ for small $x\ge 0$ and $C=C_p>0, p\in\mathbb{R}$. It follows that, for any $\varepsilon>0$ there exists $t_0=t_0(\varepsilon)>0$ such that for $t>t_0$ and $s\in(0,1)$

$$
\begin{aligned}
&\varphi_t\Big|\varphi_t I_{2t}(s)-(\kappa-\alpha)(q_t(s))^{\alpha+1}\Big| \\
&= \varphi_t^2(q_t(s))^\alpha\Big|\Big(1+\frac{q_t(s)}{\varphi_t}\Big)^{\kappa-\alpha}-1-(\kappa-\alpha)\frac{q_t(s)}{\varphi_t}\Big| \\
&\le C(q_t(s))^{\alpha+2}.
\end{aligned}
$$

Consequently, again by the dominated convergence theorem

$$
\begin{aligned}
\int_0^1 I_{2t}(s)\,ds &= \frac{\kappa-\alpha}{\varphi_t}\int_0^1 (q_t(s))^{\alpha+1}\left(1+\frac{\kappa-\alpha-1}{2}\frac{q_t(s)}{\varphi_t}(1+o(1))\right)ds \\
&= \frac{\kappa-\alpha}{\varphi_t}\int_0^1\left(D_\gamma\Big(\frac{1}{s}\Big)+H_{\gamma,\rho}\Big(\frac{1}{s}\Big)A(t)(1+o(1))\right)^{\alpha+1}\left(1+\frac{\kappa-\alpha-1}{2}\frac{q_t(s)}{\varphi_t}(1+o(1))\right)ds \\
&= \frac{\kappa-\alpha}{\varphi_t}\left(L_{\alpha+1}+A(t)M_{\alpha+1,1}(1+o(1))+\frac{\kappa-\alpha-1}{2\varphi_t}L_{\alpha+2}(1+o(1))\right),
\end{aligned} \tag{5.10}
$$

where the $o(1)$'s terms in the integral are uniform for $s\in(0,1)$, and the following are the same unless otherwise stated.

Next, we deal with $I_{3t}(s)$. First note that $|\widetilde{B}|\in \mathrm{RV}_\varsigma$. Similar arguments for (5.7) yield that, for any given $\varepsilon\in(0,1)$,

there exists some $t_0 = t_0(\varepsilon) > 0$ such that for $\varphi(t) > t_0$

$$|\widetilde{B}(\varphi_t)| \geq |\widetilde{B}((t_0-1)q_t(s))| \geq (1-\varepsilon)(q_t(s))^{\varsigma-\varepsilon}|\widetilde{B}(t_0-1)|.$$

Using further Lemma 6.1 to analyze the two cases: a) $\gamma < 0$ and b) $\gamma = 0$ above, we see that

$$\begin{aligned}
&\frac{1}{|\widetilde{B}(\varphi_t)|}\Big|\frac{I_{3t}(s)}{\widetilde{A}(\varphi_t)} - (q_t(s))^{\alpha}\left(1+\frac{q_t(s)}{\varphi_t}\right)^{\kappa-\alpha}\frac{(1/\varphi_t+1/q_t(s))^{\varrho}-1}{\varrho}\Big| \\
&\quad \leq (q_t(s))^{\alpha}\left(1+2|\kappa-\alpha|\frac{q_t(s)}{\varphi_t}\right) \\
&\quad \times \Bigg[\left(\frac{(1/\varphi_t+1/q_t(s))^{\varrho}-1}{\varrho} + \varepsilon\left(1+(q_t(s))^{-\varrho}+(q_t(s))^{-\varrho-\varsigma}\exp\left(\varepsilon\Big|\ln\left(\frac{1}{\varphi_t}+\frac{1}{q_t(s)}\right)\Big|\right)\right)\right)\mathbb{I}\{s\in\mathbb{S}_t\} \\
&\quad\quad + \frac{(q_t(s))^{\varepsilon-\varsigma}}{(1-\varepsilon)|B(t_0-1)|}\left(\frac{(q_t(s))^{\varepsilon-\varrho}}{(1-\varepsilon)|\widetilde{A}(t_0-1)|}(2t_0^{\alpha}/L(\infty)+1)+\frac{(1/\varphi_t+1/q_t(s))^{\varrho}-1}{\varrho}\right)\mathbb{I}\{s\in\mathbb{S}_t^c\}\Bigg],
\end{aligned}$$

which is integrable in $[0,1]$. Consequently, by the dominated convergence theorem

$$\begin{aligned}
\frac{1}{\widetilde{A}(\varphi_t)}\int_0^1 I_{3t}(s)\,ds &= \int_0^1 (q_t(s))^{\alpha}\left(1+\frac{q_t(s)}{\varphi_t}\right)^{\kappa-\alpha}\frac{1}{\widetilde{A}(\varphi_t)}\left(\frac{L\left(1+\frac{\varphi_t}{q_t(s)}\right)}{L(\varphi_t)}-1\right)ds \\
&= \int_0^1 \left(D_{\gamma}\left(\frac{1}{s}\right)+H_{\gamma,\rho}\left(\frac{1}{s}\right)A(t)(1+o(1))\right)^{\alpha}\left(1+(\kappa-\alpha)\frac{D_{\gamma}(1/s)}{\varphi_t}(1+o(1))\right) \\
&\quad \times \left(\frac{(q_t(s))^{-\varrho}(1+q_t(s)/\varphi_t)^{\varrho}-1}{\varrho}+\frac{(q_t(s))^{-\varrho-\varsigma}-1}{\varrho+\varsigma}\widetilde{B}(\varphi_t)(1+o(1))\right)ds \\
&= N_{\alpha,0,\varrho}+A(t)N_{\alpha,1,\varrho}(1+o(1))+\frac{(\kappa-\alpha)N_{\alpha+1,0,\varrho}+L_{\alpha-\varrho+1}}{\varphi_t}(1+o(1))+\widetilde{B}(\varphi_t)N_{\alpha,0,\varrho+\varsigma}(1+o(1))
\end{aligned}$$

which together with (5.3), (5.9) and (5.10) establishes the claim in (2.8). □

Proof of Corollary 2.5 We adopt the notation as in the proof of Theorem 2.3. A straightforward application (5.2) with $S \sim Beta(1,\kappa)$ (note that $\mathbb{P}\{S > 1-s\} = s^{\kappa}, s \in (0,1)$), we have with $\alpha = \kappa$

$$\frac{\mathbb{E}\left\{(X-U(t))_+^{\kappa}\right\}}{t^{-1}(a(t))^{\kappa}} = \int_0^1 (q_t(s))^{\kappa}\,ds = L_{\kappa} + \int_0^1 I_{1t}(s)\,ds.$$

(i) For $\gamma > 0$, it follows that $U \in 3\mathrm{ERV}_{\gamma,\rho,\eta}$ with first-, second- and third-order auxiliary functions a, A and B holds with $H_{\gamma,\rho}$ and $R_{\gamma,\rho,\eta}$ replaced by H and R given by

$$\begin{aligned}
H(x) &= \frac{x^{\gamma}}{\gamma}\frac{x^{\rho}-1}{\rho} = \frac{\gamma+\rho}{\gamma}H_{\gamma,\rho}(x)+\frac{1}{\gamma}D_{\gamma}(x) \\
R(x) &= \frac{x^{\gamma}}{\gamma}\frac{x^{\rho+\eta}-1}{\rho+\eta} = \frac{1}{\gamma}\Big(\eta(\gamma+\rho+\eta)R_{\gamma,\rho,\eta}(x)+(\gamma+\rho+\eta)H_{\gamma,\rho}(x)+D_{\gamma}(x)\Big).
\end{aligned}$$

Thus, it follows from (5.9) and Remark 6.2 that (2.10) holds. Further, note that for all $c \in \mathbb{R}$ such that $\kappa(\gamma+c) < 1$

$$\int_0^1 \big(D_{\gamma}(1/s)\big)^{\kappa} s^{-\gamma c}\,ds = \frac{1}{\gamma^{\kappa+1}}B\left(\frac{1}{\gamma}-\kappa-c,\kappa+1\right).$$

Consequently the claim follows by cumbersome calculations.

(ii) For $\gamma < 0$, we have $x_F - U \in 3\mathrm{ERV}_{\gamma,\rho,\eta}$ with first-, second- and third-order auxiliary functions a, A and B and

the limit functions $H_{\gamma,\rho}$ and $R_{\gamma,\rho,\eta}$ replaced by H and R as above. Further, for $\kappa, c > 0$

$$\int_0^1 \left(D_\gamma(1/s)\right)^{\kappa-1} s^{-\gamma c}\, ds = \frac{1}{(-\gamma)^\kappa} B\left(c - \frac{1}{\gamma}, \kappa\right).$$

Consequently, the claim follows by similar arguments. □

Proof of Theorem 3.5 It follows from Corollary 2.5 that, as $t \to \infty$

$$\begin{aligned}
&\frac{\left(\mathbb{E}\left\{(X-U(t))_+^{\kappa-1}\right\}\right)^\kappa}{\left(\mathbb{E}\left\{(X-U(t))_+^{\kappa}\right\}\right)^{\kappa-1}} \\
&\quad= \frac{1}{t}\frac{\left(L_{\kappa-1} + A(t)(M_{\kappa-1,1} + A(t)M_{\kappa-1,2} + B(t)Q_{\kappa-1})(1+o(1))\right)^\kappa}{\left(L_\kappa + A(t)(M_{\kappa,1} + A(t)M_{\kappa,2} + B(t)Q_\kappa)(1+o(1))\right)^{\kappa-1}} \\
&\quad= \frac{L_{\kappa-1}^\kappa}{tL_\kappa^{\kappa-1}}\left(1 + \left(\kappa\frac{M_{\kappa-1,1}}{L_{\kappa-1}} - (\kappa-1)\frac{M_{\kappa,1}}{L_\kappa}\right)A(t) + \left(\kappa\frac{Q_{\kappa-1}}{L_{\kappa-1}} - (\kappa-1)\frac{Q_\kappa}{L_\kappa}\right)A(t)B(t)(1+o(1))\right. \\
&\quad\quad \left. + \left(\kappa\frac{M_{\kappa-1,2}}{L_{\kappa-1}} - (\kappa-1)\frac{M_{\kappa,2}}{L_\kappa} + \frac{\kappa(\kappa-1)}{2}\left(\frac{M_{\kappa,1}}{L_\kappa} - \frac{M_{\kappa-1,1}}{L_{\kappa-1}}\right)^2\right)A^2(t)(1+o(1))\right) \\
&\quad=: \frac{\mathrm{c}}{t}\Big(1 + \Delta_\kappa A(t) + \big(\Theta_\kappa A(t)B(t) + \Lambda_\kappa A^2(t)\big)(1+o(1))\Big)
\end{aligned} \tag{5.11}$$

with

$$\begin{aligned}
&\mathrm{c} = \frac{L_{\kappa-1}^\kappa}{L_\kappa^{\kappa-1}}, \quad \Delta_\kappa = \kappa\frac{M_{\kappa-1,1}}{L_{\kappa-1}} - (\kappa-1)\frac{M_{\kappa,1}}{L_\kappa}, \quad \Theta_\kappa = \kappa\frac{Q_{\kappa-1}}{L_{\kappa-1}} - (\kappa-1)\frac{Q_\kappa}{L_\kappa} \\
&\Lambda_\kappa = \kappa\frac{M_{\kappa-1,2}}{L_{\kappa-1}} - (\kappa-1)\frac{M_{\kappa,2}}{L_\kappa} + \frac{\kappa(\kappa-1)}{2}\left(\frac{M_{\kappa,1}}{L_\kappa} - \frac{M_{\kappa-1,1}}{L_{\kappa-1}}\right)^2.
\end{aligned}$$

Note that (see also (6.2)) $A \in 2\mathrm{RV}_{\rho,\eta}$ with auxiliary function B and $|B| \in \mathrm{RV}_\eta$. We have by (3.3) and (5.11) that $(1-q)t = \mathrm{c}(1 + \Delta_\kappa A(t)(1+o(1)))$. Further, with $\epsilon_q = A(1/(1-q)), \psi_q = B(1/(1-q))$, as $q \uparrow 1$

$$\begin{aligned}
B(t) &= \mathrm{c}^\eta \psi_q(1+o(1)) \\
A(t) &= \epsilon_q\Big(c(1 + \Delta_\kappa A(t)(1+o(1)))\Big)^\rho\left(1 + \frac{\mathrm{c}^\eta - 1}{\eta}B\Big(\frac{1}{1-q}\Big)(1+o(1))\right) \\
&= \mathrm{c}^\rho\epsilon_q + \rho\mathrm{c}^{2\rho}\Delta_\kappa\epsilon_q^2(1+o(1)) + \mathrm{c}^\rho\frac{\mathrm{c}^\eta - 1}{\eta}\epsilon_q\psi_q(1+o(1))
\end{aligned} \tag{5.12}$$

Therefore, it follows by (5.11) and (5.12) that, the solution $t = t(q)$ to (3.3) has the following third-order expansion

$$t = \frac{\mathrm{c}}{1-q}\left(1 + \mathrm{c}^\rho\Delta_\kappa\epsilon_q + \mathrm{c}^{2\rho}(\rho\Delta_\kappa^2 + \Lambda_\kappa)\epsilon_q^2(1+o(1)) + \mathrm{c}^\rho\left(\mathrm{c}^\eta\Theta_\kappa + \Delta_\kappa\frac{\mathrm{c}^\eta - 1}{\eta}\right)\epsilon_q\psi_q(1+o(1))\right). \tag{5.13}$$

Thus, by (3.3), Corollary 2.5 and (5.12)

$$\begin{aligned}
&\left(\frac{\mathbb{E}\left\{(X-U(t))_+^\kappa\right\}}{1-q}\right)^{1/\kappa} = \frac{\mathbb{E}\left\{(X-U(t))_+^\kappa\right\}}{\mathbb{E}\left\{(X-U(t))_+^{\kappa-1}\right\}} \\
&\quad= a(t)\frac{L_\kappa + A(t)M_{\kappa,1} + A(t)\left(M_{\kappa,2}A(t) + Q_\kappa B(t)\right)(1+o(1))}{L_{\kappa-1} + A(t)M_{\kappa-1,1} + A(t)\left(M_{\kappa-1,2}A(t) + Q_{\kappa-1}B(t)\right)(1+o(1))} \\
&\quad= a(t)\frac{L_\kappa}{L_{\kappa-1}}\left(1 + \left(\frac{M_{\kappa,1}}{L_\kappa} - \frac{M_{\kappa-1,1}}{L_{\kappa-1}}\right)A(t) + \left(\frac{Q_\kappa}{L_\kappa} - \frac{Q_{\kappa-1}}{L_{\kappa-1}}\right)A(t)B(t)(1+o(1))\right. \\
&\quad\quad \left. + \left(\frac{M_{\kappa,2}}{L_\kappa} - \frac{M_{\kappa-1,2}}{L_{\kappa-1}} + \frac{M_{\kappa-1,1}^2}{L_{\kappa-1}^2} - \frac{M_{\kappa,1}}{L_\kappa}\frac{M_{\kappa-1,1}}{L_{\kappa-1}}\right)A^2(t)(1+o(1))\right)
\end{aligned}$$

$$= a(t)\frac{L_\kappa}{L_{\kappa-1}}\left(1+\epsilon_q \mathrm{c}^\rho \widetilde{M}_{\kappa,1}+\epsilon_q^2 \mathrm{c}^{2\rho}\left[\widetilde{M}_{\kappa,2}+\rho\widetilde{M}_{\kappa,1}\Delta_\kappa\right](1+o(1))+\epsilon_q\psi_q \mathrm{c}^\rho\left[\mathrm{c}^\eta\widetilde{Q}_\kappa+\widetilde{M}_{\kappa,1}\frac{\mathrm{c}^\eta-1}{\eta}\right](1+o(1))\right) \tag{5.14}$$

with

$$\widetilde{M}_{\kappa,1}=\frac{M_{\kappa,1}}{L_\kappa}-\frac{M_{\kappa-1,1}}{L_{\kappa-1}},\quad \widetilde{Q}_\kappa=\frac{Q_\kappa}{L_\kappa}-\frac{Q_{\kappa-1}}{L_{\kappa-1}}$$
$$\widetilde{M}_{\kappa,2}=\frac{M_{\kappa,2}}{L_\kappa}-\frac{M_{\kappa-1,2}}{L_{\kappa-1}}+\frac{M^2_{\kappa-1,1}}{L^2_{\kappa-1}}-\frac{M_{\kappa,1}}{L_\kappa}\frac{M_{\kappa-1,1}}{L_{\kappa-1}}.$$

We will present next the proof for $\gamma>0$ and $\rho,\eta<0$. The other cases follow by similar arguments and thus are omitted here.

Since $U\in 3\mathrm{RV}_{\gamma,\rho,\eta}$ with auxiliary functions A and B, we have $U\in 3\mathrm{ERV}_{\gamma,\rho,\eta}$ with first-, second- and third-order auxiliary functions $a(t)=\gamma U(t), A(t)$ and $B(t)$. By Lemma 6.1 and Remark 6.2, we have using (5.13) with $t=t_q$, as $q\uparrow 1$

$$\begin{aligned}
&\frac{U(t)}{U(1/(1-q))}\\
&=((1-q)t)^\gamma+\epsilon_q((1-q)t)^\gamma\frac{((1-q)t)^\rho-1}{\rho}+\epsilon_q\psi_q((1-q)t)^\gamma\frac{((1-q)t)^{\rho+\eta}-1}{\rho+\eta}(1+o(1))\\
&=\mathrm{c}^\gamma\left(1+\gamma\mathrm{c}^\rho\Delta_\kappa\epsilon_q+\gamma\mathrm{c}^{2\rho}\left(\Lambda_\kappa+\left[\rho+\frac{\gamma-1}{2}\right]\Delta_\kappa^2\right)\epsilon_q^2(1+o(1))+\gamma\mathrm{c}^\rho\left(\mathrm{c}^\eta\Theta_\kappa+\Delta_\kappa\frac{\mathrm{c}^\eta-1}{\eta}\right)\epsilon_q\psi_q(1+o(1))\right)\\
&\quad+\epsilon_q\mathrm{c}^\gamma\frac{\mathrm{c}^\rho-1}{\rho}\left(1+\left[\gamma\mathrm{c}^\rho\Delta_\kappa+\frac{\rho\mathrm{c}^{2\rho}}{\mathrm{c}^\rho-1}\Delta_\kappa\right]\epsilon_q(1+o(1))\right)+\epsilon_q\psi_q\mathrm{c}^\gamma\frac{\mathrm{c}^{\rho+\eta}-1}{\rho+\eta}(1+o(1))\\
&=\mathrm{c}^\gamma\left(1+\epsilon_q\left[\gamma\mathrm{c}^\rho\Delta_\kappa+\frac{\mathrm{c}^\rho-1}{\rho}\right]+\epsilon_q\psi_q\left[\gamma\mathrm{c}^\rho\left(\mathrm{c}^\eta\Theta_\kappa+\Delta_\kappa\frac{\mathrm{c}^\eta-1}{\eta}\right)+\frac{\mathrm{c}^{\rho+\eta}-1}{\rho+\eta}\right](1+o(1))\right.\\
&\quad\left.+\epsilon_q^2\left[\gamma\mathrm{c}^{2\rho}\left(\Lambda_\kappa+\left[\rho+\frac{\gamma-1}{2}\right]\Delta_\kappa^2\right)+\frac{\mathrm{c}^\rho-1}{\rho}\left(\gamma\mathrm{c}^\rho\Delta_\kappa+\frac{\rho\mathrm{c}^{2\rho}}{\mathrm{c}^\rho-1}\Delta_\kappa\right)\right](1+o(1))\right),
\end{aligned}\tag{5.15}$$

which together with (5.14) yields that (recall $a(t)=\gamma U(t)$)

$$\begin{aligned}
H_q[X] &= U(t)+\Big(\frac{\mathbb{E}\left\{(X-U(t))_+^\kappa\right\}}{1-q}\Big)^{1/\kappa}\\
&= \mathrm{c}^\gamma F^{\leftarrow}(q)\left(1+\frac{\gamma L_\kappa}{L_{\kappa-1}}\right)\left(1+\epsilon_q\frac{\gamma L_\kappa}{L_{\kappa-1}}\left(1+\frac{\gamma L_\kappa}{L_{\kappa-1}}\right)^{-1}\right.\\
&\quad\times\left(\mathrm{c}^\rho\widetilde{M}_{\kappa,1}+\epsilon_q\mathrm{c}^{2\rho}[\widetilde{M}_{\kappa,2}+\rho\widetilde{M}_{\kappa,1}\Delta_\kappa](1+o(1))+\psi_q\mathrm{c}^\rho\left[\mathrm{c}^\eta\widetilde{Q}_\kappa+\widetilde{M}_{\kappa,1}\frac{\mathrm{c}^\eta-1}{\eta}\right](1+o(1))\right)\bigg)\\
&\quad\times\left(1+\epsilon_q\left[\gamma\mathrm{c}^\rho\Delta_\kappa+\frac{\mathrm{c}^\rho-1}{\rho}\right]+\epsilon_q\psi_q\left[\gamma\mathrm{c}^\rho\left(\mathrm{c}^\eta\Theta_\kappa+\Delta_\kappa\frac{\mathrm{c}^\eta-1}{\eta}\right)+\frac{\mathrm{c}^{\rho+\eta}-1}{\rho+\eta}\right](1+o(1))\right.\\
&\quad\left.+\epsilon_q^2\left[\gamma\mathrm{c}^{2\rho}\left(\Lambda_\kappa+\left[\rho+\frac{\gamma-1}{2}\right]\Delta_\kappa^2\right)+\frac{\mathrm{c}^\rho-1}{\rho}\left(\gamma\mathrm{c}^\rho\Delta_\kappa+\frac{\rho\mathrm{c}^{2\rho}}{\mathrm{c}^\rho-1}\Delta_\kappa\right)\right](1+o(1))\right)\\
&=: c_0F^{\leftarrow}(q)\Big(1+c_1\epsilon_q+c_2\epsilon_q^2(1+o(1))+c_3\epsilon_q\psi_q(1+o(1))\Big),
\end{aligned}\tag{5.16}$$

which together with $L_\kappa/L_{\kappa-1}=\kappa/(1-\kappa\gamma)$ implies that

$$c_0 = \frac{\mathrm{c}^\gamma}{1-\kappa\gamma},\quad c_1=\gamma\mathrm{c}^\rho(\kappa\widetilde{M}_{\kappa,1}+\Delta_\kappa)+\frac{\mathrm{c}^\rho-1}{\rho}$$

$$
\begin{aligned}
c_2 &= \gamma \mathrm{c}^{2\rho}[\kappa \widetilde{M}_{\kappa,2} + \rho\kappa \widetilde{M}_{\kappa,1}\Delta_\kappa] + \gamma \mathrm{c}^{\rho}\kappa \widetilde{M}_{\kappa,1}\left[\gamma \mathrm{c}^{\rho}\Delta_\kappa + \frac{\mathrm{c}^{\rho}-1}{\rho}\right] \\
&\quad + \gamma \mathrm{c}^{2\rho}\left(\Lambda_\kappa + \left[\rho + \frac{\gamma-1}{2}\right]\Delta_\kappa^2\right) + \frac{\mathrm{c}^{\rho}-1}{\rho}\left(\gamma \mathrm{c}^{\rho}\Delta_\kappa + \frac{\rho \mathrm{c}^{2\rho}}{\mathrm{c}^{\rho}-1}\Delta_\kappa\right) \\
c_3 &= \kappa\gamma \mathrm{c}^{\rho}\left[\mathrm{c}^{\eta}\widetilde{Q}_\kappa + \widetilde{M}_{\kappa,1}\frac{\mathrm{c}^{\eta}-1}{\eta}\right] + \gamma \mathrm{c}^{\rho}\left(\mathrm{c}^{\eta}\Theta_\kappa + \Delta_\kappa \frac{\mathrm{c}^{\eta}-1}{\eta}\right) + \frac{\mathrm{c}^{\rho+\eta}-1}{\rho+\eta}
\end{aligned}
$$

Next, we will calculate the four coefficients $c_0, \dots, c_3$ in turn.

Recalling that $\mathrm{c} = L_{\kappa-1}^{\kappa}/L_{\kappa}^{\kappa-1}$, we have

$$
\mathrm{c} = \kappa\Big(\frac{1-\kappa\gamma}{\kappa|\gamma|}\Big)^{\kappa}\xi_{\kappa,0}, \quad \xi_{\kappa,\rho} = \frac{(\kappa-1)|\gamma|}{1-\rho-\kappa\gamma}\xi_{\kappa-1,\rho}.
$$

Further, by (5.11) and (5.14) (recall the symmetry of $(\widetilde{M}_{\kappa,1}, \Delta_\kappa)$ and $(\widetilde{Q}_\kappa, \Theta_\kappa)$), we have that (2.9) holds and further

$$
\begin{cases}
\kappa \widetilde{M}_{\kappa,1} + \Delta_\kappa = \frac{M_{\kappa,1}}{L_\kappa} = \frac{1}{\gamma\rho}\left(\frac{\xi_{\kappa,\rho}}{\xi_{\kappa,0}} - 1\right) \\
\kappa \widetilde{Q}_\kappa + \Theta_\kappa = \frac{Q_\kappa}{L_\kappa} = \frac{1}{\gamma(\rho+\eta)}\left(\frac{\xi_{\kappa,\rho+\eta}}{\xi_{\kappa,0}} - 1\right) \\
\widetilde{Q}_\kappa = \frac{1}{\gamma(\rho+\eta)}\left(\frac{\xi_{\kappa,\rho+\eta}}{\xi_{\kappa,0}} - \frac{\xi_{\kappa-1,\rho+\eta}}{\xi_{\kappa-1,0}}\right) \\
\frac{M_{\kappa,2}}{L_\kappa} = \frac{1}{2\gamma^2\rho^2}\left((1-\gamma-2\rho)\frac{\xi_{\kappa,2\rho}}{\xi_{\kappa,0}} - 2(1-\gamma-\rho)\frac{\xi_{\kappa,\rho}}{\xi_{\kappa,0}} + 1 - \gamma\right).
\end{cases}
\tag{5.17}
$$

Therefore

$$
\Theta_\kappa = \frac{1}{\gamma(\rho+\eta)}\left(\kappa\frac{\xi_{\kappa-1,\rho+\eta}}{\xi_{\kappa-1,0}} - (\kappa-1)\frac{\xi_{\kappa,\rho+\eta}}{\xi_{\kappa,0}} - 1\right).
$$

Hence, by (5.17)

$$
c_1 = \mathrm{c}^{\rho}\left(\gamma\frac{M_{\kappa,1}}{L_\kappa} + \frac{1}{\rho}\right) - \frac{1}{\rho} = \frac{1}{\rho}\left(\mathrm{c}^{\rho}\frac{\xi_{\kappa,\rho}}{\xi_{\kappa,0}} - 1\right).
$$

Moreover, we rewrite c_2 as follows.

$$
c_2 = \gamma \mathrm{c}^{2\rho}\left[\kappa \widetilde{M}_{\kappa,2} + \Lambda_\kappa + (\gamma+\rho)\kappa \widetilde{M}_{\kappa,1}\Delta_\kappa + \left(\rho + \frac{\gamma-1}{2}\right)\Delta_\kappa^2 + \frac{1}{\gamma}\Delta_\kappa\right] + \gamma \mathrm{c}^{\rho}\frac{\mathrm{c}^{\rho}-1}{\rho}(\kappa \widetilde{M}_{\kappa,1} + \Delta_\kappa).
$$

Note further by (5.11) and (5.14) that

$$
\kappa \widetilde{M}_{\kappa,2} + \Lambda_\kappa = \frac{M_{\kappa,2}}{L_\kappa} + \kappa \widetilde{M}_{\kappa,1}\left(\frac{\kappa-1}{2}\widetilde{M}_{\kappa,1} - \frac{M_{\kappa-1,1}}{L_{\kappa-1}}\right).
$$

The claim for c_2 follows by (5.17).

Finally, it follows again by (5.17) that

$$
\begin{aligned}
c_3 &= \mathrm{c}^{\rho+\eta}\left[\gamma\kappa\widetilde{Q}_\kappa + \gamma\Theta_\kappa + \frac{1}{\rho+\eta}\right] - \frac{1}{\rho+\eta} + \gamma \mathrm{c}^{\rho}\frac{\mathrm{c}^{\eta}-1}{\eta}\frac{M_{\kappa,1}}{L_\kappa} \\
&= \frac{1}{\rho+\eta}\left(\mathrm{c}^{\rho+\eta}\frac{\xi_{\kappa,\rho+\eta}}{\xi_{\kappa,0}} - 1\right) + \frac{\mathrm{c}^{\rho}(\mathrm{c}^{\eta}-1)}{\rho\eta}\left(\frac{\xi_{\kappa,\rho}}{\xi_{\kappa,0}} - 1\right).
\end{aligned}
$$

Now for $\gamma < 0$, note that $a(t) = -\gamma(x_F - U(t))$ and $x_F - U(t) \in 3\mathrm{RV}_{\gamma,\rho,\eta}$. It follows that (5.15) holds for $(x_F - U(t))/(x_F - U(1/(1-q)))$. Further (5.16) holds by replacing $H_q[X]$ and $F^{\leftarrow}(q)$ by $x_F - H_q[X]$ and $x_F - F^{\leftarrow}(q)$, respectively. The remaing arguments are the same as for $\gamma > 0$. This completes the proof of Theorem 3.5. □

Proof of Theorem 3.7 Since $\alpha := 1/\gamma > 1$, we have $\mathbb{E}\{X\} < \infty$ and thus the expectile e_q satisfies (3.5). It follows by Theorem 2.3.9 in [10] that $\overline{F} \in 2\mathrm{RV}_{-\alpha,\alpha\rho}$ with auxiliary function $A^*(x) = \alpha^2 A(1/\overline{F}(x))$. By Corollary 4.4 in [24]

$$\mathbb{E}\{(X-x)_+\} = \frac{x\overline{F}(x)}{\alpha-1}\left(1+\frac{A^*(x)}{\alpha-1-\alpha\rho}(1+o(1))\right), \quad x\to\infty,$$

which together with the first-order approximation $e_q = (\alpha-1)^{-1/\alpha}F^{\leftarrow}(q)(1+o(1)) \to \infty, q \uparrow 1$ (see Proposition 2.3 in [2]) yields that

$$\begin{aligned}
e_q &= \frac{1}{1-\mathbb{E}\{X\}/e_q}\frac{2q-1}{1-q}\mathbb{E}\{(X-e_q)_+\} \\
&= \frac{e_q}{1-q}\frac{\overline{F}(e_q)}{\alpha-1}(1-2(1-q))\left(1+\frac{A^*(e_q)}{\alpha-1-\alpha\rho}(1+o(1))+\frac{(\alpha-1)^{1/\alpha}\mathbb{E}\{X\}}{F^{\leftarrow}(q)}(1+o(1))\right) \\
&= \frac{e_q}{1-q}\frac{\overline{F}(e_q)}{\alpha-1}\left(1+\frac{\alpha d_0}{F^{\leftarrow}(q)}(1+o(1))+\alpha d_1(1-q)(1+o(1))+\alpha D\epsilon_q(1+o(1))\right). \qquad (5.18)
\end{aligned}$$

The last step follows since $|A| \in \mathrm{RV}_\rho$ and $(1-q)/\overline{F}(e_q) = 1/(\alpha-1)(1+o(1))$.

Further by $U \in 2\mathrm{RV}_{\gamma,\rho}$ with auxiliary function A

$$\begin{aligned}
e_q &= U\left(\frac{1}{1-q}\frac{1}{\alpha-1}\left(1+\frac{\alpha d_0}{F^{\leftarrow}(q)}(1+o(1))+\alpha d_1(1-q)(1+o(1))+\alpha D\epsilon_q(1+o(1))\right)\right) \\
&= \overline{F}^{\leftarrow}(q)\left[\frac{1}{\alpha-1}\left(1+\frac{\alpha d_0}{F^{\leftarrow}(q)}(1+o(1))+\alpha d_1(1-q)(1+o(1))+\alpha D\epsilon_q(1+o(1))\right)\right]^{1/\alpha} \\
&\quad \times\left[1+\frac{(\alpha-1)^{-\rho}-1}{\rho}\epsilon_q(1+o(1))\right] \\
&= (\alpha-1)^{-1/\alpha}F^{\leftarrow}(q)\left(1+\frac{d_0}{F^{\leftarrow}(q)}(1+o(1))+d_1(1-q)(1+o(1))+d_2\epsilon_q(1+o(1))\right)
\end{aligned}$$

establishing the proof of the first claim.

First by Lemma 6.1

$$\mathbb{E}\{(X-x)_+\} = \frac{x\overline{F}(x)}{\alpha-1}\left(1+\frac{A^*(x)}{\alpha-1-\alpha\rho}\left(1+\frac{\alpha-1-\alpha\rho}{\alpha-1-\alpha(\rho+\eta)}B^*(x)(1+o(1))\right)\right)$$

with $A^*(x) = \alpha^2 A(1/\overline{F}(x)), B^*(x) = B(1/\overline{F}(x))$. Therefore

$$\begin{aligned}
\frac{1}{\overline{F}(e_q)} &= \frac{1-2(1-q)}{1-q}\frac{1+\mathbb{E}\{X\}/e_q+(\mathbb{E}\{X\}/e_q)^2(1+o(1))}{\alpha-1} \\
&\quad\times\left[1+\frac{A^*(e_q)}{\alpha-1-\alpha\rho}\left(1+\frac{\alpha-1-\alpha\rho}{\alpha-1-\alpha(\rho+\eta)}B^*(e_q)(1+o(1))\right)\right] \\
&= \frac{1}{1-q}\frac{1}{\alpha-1}\left[1-2(1-q)+\frac{\mathbb{E}\{X\}}{e_q}+\frac{A^*(e_q)}{\alpha-1-\alpha\rho}\right. \\
&\quad -2(1-q)\frac{\mathbb{E}\{X\}}{e_q}(1+o(1))-2(1-q)\frac{A^*(e_q)}{\alpha-1-\alpha\rho}(1+o(1))+\frac{A^*(e_q)}{\alpha-1-\alpha\rho}\frac{\mathbb{E}\{X\}}{e_q}(1+o(1)) \\
&\quad \left.+\Big(\frac{\mathbb{E}\{X\}}{e_q}\Big)^2(1+o(1))+\frac{A^*(e_q)B^*(e_q)}{\alpha-1-\alpha(\rho+\eta)}(1+o(1))\right] \\
&=: \frac{x_q}{1-q}.
\end{aligned}$$

Noting that $A \in 2\mathrm{RV}_{\rho,\eta}$ with auxiliary function B, and $|B| \in \mathrm{RV}_\eta$, it follows by (5.18) that

$$\begin{aligned}\frac{A^*(e_q)}{\alpha-1-\alpha\rho} &= \frac{\alpha^2}{\alpha-1-\alpha\rho}\epsilon_q x_q^\rho\left(1+\frac{x_q^\eta-1}{\eta}\psi_q(1+o(1))\right) = \alpha D\epsilon_q\left[1-2\rho(1-q)(1+o(1))\right.\\ &\quad \left.+\frac{\alpha\rho d_0}{F^{\leftarrow}(q)}(1+o(1))+\alpha\rho D\epsilon_q(1+o(1))+\frac{(\alpha-1)^{-\eta}-1}{\eta}\psi_q(1+o(1))\right]\\ B^*(e_q) &= (\alpha-1)^{-\eta}\psi_q(1+o(1)),\end{aligned}$$

which together with Theorem 3.5 implies that

$$\begin{aligned}x_q &= \frac{1}{\alpha-1}\left[1-2(1-q)+\frac{\alpha d_0}{F^{\leftarrow}(q)}\left(1-\frac{d_0}{F^{\leftarrow}(q)}(1+o(1))-d_1(1-q)(1+o(1))-d_2\epsilon_q(1+o(1))\right)\right.\\ &\quad +\alpha D\epsilon_q\left(1-2\rho(1-q)(1+o(1))+\frac{\alpha\rho d_0}{F^{\leftarrow}(q)}(1+o(1))+\alpha\rho D\epsilon_q(1+o(1))+\frac{(\alpha-1)^{-\eta}-1}{\eta}\psi_q(1+o(1))\right)\\ &\quad \left.-\alpha\left(2d_0\frac{1-q}{F^{\leftarrow}(q)}+2D(1-q)\epsilon_q-\alpha Dd_0\frac{\epsilon_q}{F^{\leftarrow}(q)}-\frac{\alpha d_0^2}{(F^{\leftarrow}(q))^2}-\frac{\alpha(\alpha-1)^{-(\rho+\eta)}}{\alpha-1-\alpha(\rho+\eta)}\epsilon_q\psi_q\right)(1+o(1))\right]. \quad (5.19)\end{aligned}$$

Next, by $U \in 3\mathrm{RV}_{\gamma,\rho,\eta}$ with auxiliary functions A and B

$$\frac{e_q}{F^{\leftarrow}(q)} = \frac{U(x_q/(1-q))}{U(1/(1-q))} = x_q^{1/\alpha}\left(1+\frac{x_q^\rho-1}{\rho}\epsilon_q+\frac{x_q^{\rho+\eta}-1}{\rho+\eta}\epsilon_q\psi_q(1+o(1))\right), \quad (5.20)$$

with (recall x_q defined in (5.19))

$$\begin{aligned}x_q^{1/\alpha} &= (\alpha-1)^{-1/\alpha}\left[1+d_1(1-q)+\frac{d_0}{F^{\leftarrow}(q)}\left(1-\frac{d_0}{F^{\leftarrow}(q)}(1+o(1))-d_1(1-q)(1+o(1))-d_2\epsilon_q(1+o(1))\right)\right.\\ &\quad +D\epsilon_q\left(1-2\rho(1-q)(1+o(1))+\frac{\alpha\rho d_0}{F^{\leftarrow}(q)}(1+o(1))+\alpha\rho D\epsilon_q(1+o(1))+\frac{(\alpha-1)^{-\eta}-1}{\eta}\psi_q(1+o(1))\right)\\ &\quad -\left(2d_0\frac{1-q}{F^{\leftarrow}(q)}+2D(1-q)\epsilon_q-\alpha Dd_0\frac{\epsilon_q}{F^{\leftarrow}(q)}-\frac{\alpha d_0^2}{(F^{\leftarrow}(q))^2}-\frac{\alpha(\alpha-1)^{-(\rho+\eta)}}{\alpha-1-\alpha(\rho+\eta)}\epsilon_q\psi_q\right)(1+o(1))\\ &\quad \left.+\frac{1-\alpha}{2}\left(\frac{d_0^2}{(F^{\leftarrow}(q))^2}+d_1^2(1-q)^2+D^2\epsilon_q^2\right)(1+o(1))\right]\\ \frac{x_q^\rho-1}{\rho} &= \frac{1}{\rho}\left((\alpha-1)^{-\rho}\left[1+\alpha\rho d_1(1-q)(1+o(1))+\frac{\alpha\rho d_0}{F^{\leftarrow}(q)}(1+o(1))+\alpha\rho D\epsilon_q(1+o(1))\right]-1\right)\\ \frac{x_q^{\rho+\eta}-1}{\rho+\eta} &= \frac{(\alpha-1)^{-(\rho+\eta)}-1}{\rho+\eta}(1+o(1)).\end{aligned}$$

Consequently, the desired result follows by (5.20) and elementary calculations. □

Proof of Theorem 3.9 By Corollary 4.4 in [24]

$$\mathbb{E}\{(X-x)_+\} = \frac{x_F-x}{\alpha+1}\overline{F}(x)\left(1-\frac{\alpha^2}{\alpha+1-\alpha\rho}A\Big(\frac{1}{\overline{F}(x)}\Big)(1+o(1))\right), \quad x \uparrow x_F.$$

Further by (3.5)

$$e_q-\mathbb{E}\{X\} = \frac{x_F-e_q}{1-q}\frac{\overline{F}(e_q)}{\alpha+1}\left(1-2(1-q)(1+o(1))-\frac{\alpha^2}{\alpha+1-\alpha\rho}A\Big(\frac{1}{\overline{F}(e_q)}\Big)(1+o(1))\right).$$

It follows from Proposition 2.5 in [2] that $x_F-e_q = \widetilde{C}(1-q)^{1/(\alpha+1)} \to 0$ for some positive constant $\widetilde{C}$. We have thus $t_q := (x_F-e_q)/(1-q) \to \infty$. By (3.5) and Taylor's expansion $1/(1-x) = 1+x(1+o(1))$, $x \to 0$

$$\frac{1}{\overline{F}(e_q)} = \frac{x_F-e_q}{1-q}\frac{1}{(\alpha+1)(x_F-\mathbb{E}\{X\})}$$

$$
\begin{aligned}
& \times\left(1-\frac{\alpha^2}{\alpha+1-\alpha\rho}A\Big(\frac{1}{\overline{F}(e_q)}\Big)(1+o(1))+\frac{x_F-e_q}{x_F-\mathbb{E}\{X\}}(1+o(1))\right)\\
=: \; & \frac{t_q}{x_0}\left(1-\frac{\alpha^2}{\alpha+1-\alpha\rho}A\Big(\frac{1}{\overline{F}(e_q)}\Big)(1+o(1))+\frac{x_F-e_q}{x_F-\mathbb{E}\{X\}}(1+o(1))\right).
\end{aligned}
$$

Noting further that $x_F - U \in 2\mathrm{RV}_{\gamma,\rho}$ with auxiliary function A and $|A| \in \mathrm{RV}_\rho$

$$
\begin{aligned}
x_F-e_q &= (x_F-U(t_q))x_0^{1/\alpha}\left(1+\frac{x_0^{-\rho}-1}{\rho}A(t_q)(1+o(1))\right)\\
&\quad\times\left(1+\frac{\alpha}{\alpha+1-\alpha\rho}A\Big(\frac{1}{\overline{F}(e_q)}\Big)(1+o(1))+\frac{\gamma(x_F-e_q)}{x_F-\mathbb{E}\{X\}}(1+o(1))\right)\\
&= C\Big(\frac{x_0}{t_q}\Big)^{1/\alpha}\left(1+\frac{x_0^{-\rho}}{\rho}A(t_q)(1+o(1))+\frac{\alpha}{\alpha+1-\alpha\rho}A\Big(\frac{1}{\overline{F}(e_q)}\Big)(1+o(1))+\frac{\gamma(x_F-e_q)}{x_F-\mathbb{E}\{X\}}(1+o(1))\right)\\
&= C\Big(\frac{x_0}{t_q}\Big)^{1/\alpha}\left(1+\frac{(\alpha+1)x_0^{-\rho}}{\rho(\alpha+1-\alpha\rho)}A(t_q)(1+o(1))-\frac{x_F-e_q}{\alpha(x_F-\mathbb{E}\{X\})}(1+o(1))\right). \qquad (5.21)
\end{aligned}
$$

Clearly, $x_F - e_q = C(x_0/t_q)^{1/\alpha}(1+o(1))$ we have (recall $t_q := (x_F - e_q)/(1-q)$)

$$
x_F-e_q=(C^\alpha x_0(1-q))^{1/(\alpha+1)}(1+o(1))
$$

and thus

$$
t_q=\Big(\frac{C}{1-q}\Big)^{\alpha/(\alpha+1)}x_0^{1/(\alpha+1)}(1+o(1)).
$$

Consequently, by (5.21)

$$
\begin{aligned}
x_F-e_q &= (C^\alpha x_0(1-q))^{1/(\alpha+1)}\left(1-\frac{(C^\alpha x_0(1-q))^{1/(\alpha+1)}}{\alpha(x_F-\mathbb{E}\{X\})}(1+o(1))\right.\\
&\qquad\left.+\frac{(\alpha+1)(C/x_0)^{\alpha\rho/(\alpha+1)}}{\rho(\alpha+1-\alpha\rho)}A\left((1-q)^{-\frac{\alpha}{\alpha+1}}\right)(1+o(1))\right).
\end{aligned}
$$

We obtain the desired result. □

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6 Appendix

In this appendix we first establish an *extensional Drees' type inequality* in Lemma 6.1 for the third-order extended regularly varying functions. Then we present a proposition concerning the third-order regular variation properties under generalized inverse transformations. Recall that $D_\gamma, H_{\gamma,\rho}$ and $R_{\gamma,\rho,\eta}$ are defined by (2.3).

Lemma 6.1. *If $f \in 3\mathrm{ERV}_{\gamma,\rho,\eta}$ with auxiliary functions a, A and B, then for any $\epsilon > 0$, there exists $t_0 = t_0(\varepsilon) > 0, C > 0$ such that for all $\min(t, tx) \geq t_0$*

$$\begin{aligned}&\Big|\frac{f(tx) - f(t) - a_0(t)D_\gamma(x) - a_1(t)H_{\gamma,\rho}(x)}{a_2(t)} - R_{\gamma,\rho,\eta}(x)\Big| \\ &\quad \leq \epsilon(1 + x^\gamma + 2x^{\gamma+\rho} + 4x^{\gamma+\rho+\eta}e^{\epsilon|\ln x|} + \mathbb{I}\{\gamma = \rho = 0\}e^{C|\ln x|}) \end{aligned} \tag{6.1}$$

with $a_0(t) = a(t), a_1(t) = a_0(t)A(t)$ and $a_2(t) = a_0(t)A(t)B(t)$.

Remark 6.2. *a) We see that (6.1) also holds for $f \in 3\mathrm{RV}_{\gamma,\rho,\eta}$ with $H_{\gamma,\rho}$ and $R_{\gamma,\rho,\eta}$ replaced respectively by H and R given by*

$$H(x) = c_1 H_{\gamma,\rho}(x) + c_2 D_\gamma(x), \quad R(x) = d_1 R_{\gamma,\rho,\eta}(x) + d_2 H_{\gamma,\rho}(x) + d_3 D_\gamma(x)$$

with $c_i, d_j \in \mathbb{R}, i = 1, 2, j = 1, 2, 3$.

b) The inequality (6.1) is the third-order form of Lemma 5.2 in [13], which is the so-called the extensional Drees' inequality*, which is different from those by Theorem 2.1 in [15] and Lemma 2.1 in [36]*

Proof of Lemma 6.1 For simplicity, we denote

$$\begin{aligned} I_0(t,x) &= \frac{a_0(tx) - a_0(t)x^\gamma - a_1(t)x^\gamma D_\rho(x)}{a_2(t)}, \quad I_1(t,x) = \frac{a_1(tx) - a_1(t)x^{\gamma+\rho}}{a_2(t)} \\ I_2(t,x) &= \frac{a_2(tx)}{a_2(t)}, \quad I(t,x) = \frac{f(tx) - f(t) - a_0(t)D_\gamma(x) - a_1(t)H_{\gamma,\rho}(x)}{a_2(t)}. \end{aligned}$$

It follows from Theorem 2.1 in [15] that

$$\lim_{t\to\infty} I_0(t,x) = x^\gamma H_{\rho,\eta}(x), \quad \lim_{t\to\infty} I_1(t,x) = x^{\gamma+\rho} D_\eta(x), \quad \lim_{t\to\infty} I_2(t,x) = x^{\gamma+\rho+\eta}. \tag{6.2}$$

Next, we will consider the following four cases: Case a: $\gamma \neq 0$ and $\gamma + \rho \neq 0$; Case b: $\gamma \neq 0$ and $\gamma + \rho = 0$; Case c: $\gamma = 0$ and $\gamma + \rho \neq 0$ and Case d: $\gamma = \rho = 0$, respectively.

Case a: $\gamma \neq 0$ and $\gamma + \rho \neq 0$. Let $g(t) = f(t) - a_0(t)/\gamma + a_1(t)/(\gamma(\gamma + \rho))$. It follows from (2.3) and (6.2) that, for all $x > 0$

$$\begin{aligned} \frac{g(tx) - g(t)}{a_2(t)/(\gamma(\gamma+\rho))} &= \gamma(\gamma+\rho)I(t,x) - (\gamma+\rho)I_0(t,x) + I_1(t,x) \\ &\to \gamma(\gamma+\rho)R_{\gamma,\rho,\eta}(x) - (\gamma+\rho)x^\gamma H_{\rho,\eta}(x) + x^{\gamma+\rho}D_\eta(x) \\ &= D_{\gamma+\rho+\eta}(x), \quad t\to\infty. \end{aligned}$$

Hence, it follows from Lemma 5.2 in [13] and similar arguments of Lemma 2.1 in [11] that, for any $\epsilon > 0$, there exists some $t_0 = t_0(\varepsilon) > 0$ such that for all $\min(t, tx) \geq t_0$

$$\Big|\frac{g(tx) - g(t)}{a_2(t)/(\gamma(\gamma+\rho))} - D_{\gamma+\rho+\eta}(x)\Big| \leq \epsilon(1 + x^{\gamma+\rho+\eta}e^{\epsilon|\ln x|})$$

$$\left|x^{-(\gamma+\rho)}I_1(t,x) - D_\eta(x)\right| = \left|\frac{(tx)^{-(\gamma+\rho)}a_1(tx) - t^{-(\gamma+\rho)}a_1(t)}{t^{-(\gamma+\rho)}a_2(t)} - D_\eta(x)\right| \leq \epsilon(1 + x^\eta e^{\epsilon|\ln x|})$$

and

$$\left|x^{-\gamma}I_0(t,x) - H_{\rho,\eta}(x)\right| = \left|\frac{(tx)^{-\gamma}a_0(tx) - t^{-\gamma}a_0(t) - t^{-\gamma}a_1(t)D_\rho(x)}{t^{-\gamma}a_2(t)} - H_{\rho,\eta}(x)\right| \leq \epsilon(1 + x^\rho + 2x^{\rho+\eta}e^{\epsilon|\ln x|}).$$

Consequently

$$\left|I(t,x) - R_{\gamma,\rho,\eta}\right| \leq \epsilon\left(\frac{1 + x^{\gamma+\rho+\eta}e^{\epsilon|\ln x|}}{\gamma(\gamma+\rho)} + \frac{x^\gamma(1 + x^\rho + 2x^{\rho+\eta}e^{\epsilon|\ln x|})}{\gamma} + \frac{x^{\gamma+\rho}(1 + x^\eta e^{\epsilon|\ln x|})}{\gamma(\gamma+\rho)}\right)$$

establishing our proof for Case a.

Case b: $\gamma \neq 0$ and $\gamma + \rho = 0$. Letting $g(t) = f(t) - a_0(t)/\gamma$, we have

$$\begin{aligned}&\frac{g(tx) - g(t) - (-a_1(t)/\gamma)\ln x}{a_2(t)/\rho}\\ &= \rho I(t,x) + I_0(t,x)\\ &\to \rho R_{\gamma,\rho,\eta}(x) + x^\gamma H_{\rho,\eta}(x)\\ &= H_{0,\eta}(x), \quad t \to \infty.\end{aligned}$$

Consequently, the claim follows by similar arguments for Case a.

Case c: $\gamma = 0$ and $\gamma + \rho \neq 0$. Let $g(t) = f(t) - (a_1(t) - a_2(t)/(\rho+\eta))/\rho^2$, then

$$\begin{aligned}&\frac{g(tx) - g(t) - (a_0(t) - a_1(t)/\rho)\ln x}{a_2(t)/(-\rho)}\\ &= (-\rho)I(t,x) + \frac{1}{\rho}I_1(t,x) - \frac{1}{\rho(\rho+\eta)}(I_2(t,x) - 1)\\ &\to (-\rho)\left(R_{0,\rho,\eta}(x) - \frac{x^\rho}{\rho^2}D_\eta(x) + \frac{1}{\rho^2}D_{\rho+\eta}(x)\right)\\ &= H_{0,\rho+\eta}(x), \quad t \to \infty.\end{aligned}$$

The remaining proof is similar to those for Case a and thus is omitted here.

Case d: $\gamma = \rho = 0$. We first consider that $\eta < 0$. Since $(a_1(tx) - a_1(t))/a_2(t) \to D_\eta(x)$, we have by Theorem 1.10 in [16] that, there exists some constant $c \neq 0$ such that

$$\lim_{t\to\infty} a_1(t) = c, \quad \lim_{t\to\infty}\frac{a_1(t) - c}{a_2(t)} = \frac{1}{\eta}. \tag{6.3}$$

Letting $g(t) = f(t) - c(\ln t)^2/2$, we have

$$\frac{g(tx) - g(t) - (a_0(t) - c\ln t)\ln x}{a_1(t) - c} = \frac{a_2(t)}{a_1(t) - c}\left(I(t,x) - R_{0,0,\eta}(x) + \left(\frac{a_1(t) - c}{a_2(t)} - \frac{1}{\eta}\right)\frac{\ln^2 x}{2} + \frac{1}{\eta}H_{0,\eta}(x)\right)$$

$$\to \quad H_{0,\eta}(x).$$

We see from (2.3) that $g \in 2\mathrm{ERV}_{0,\eta}$ with auxiliary function $a_0(t) - c\ln t$ and $a_1(t) - c$. Noting further that there exist two constants $C, D > 0$ such that for all $x \in (0,\infty)$ (cf. Lemma 2.2 in [36])

$$|H_{0,\tau}(x)| \le D\exp\left(C|\ln x|\right), \quad \tau \le 0. \tag{6.4}$$

The claim follows by (6.3) and similar arguments as for Case a.

Next, we deal with the case $\eta = 0$. Letting $g(t) = f(t) - \int_1^t a_0(u)/u\,du + a_1(t)$, we have by (2.3) and (6.2)

$$\begin{aligned}\frac{g(tx)-g(t)}{a_2(t)} &= I(t,x) - \int_1^x \frac{I_0(t,u)}{u}\,du + I_1(t,x)\\ &\to \frac{(\ln x)^3}{6} - \int_1^x \frac{(\ln u)^2}{2u}\,du + \ln x = \ln x,\end{aligned}$$

where we used in the second step the dominated convergence theorem (see e.g. Lemma 5.2 in [13]). The claim follows by(6.4) and the same arguments for Case a. Consequently, we complete the proof. □

Proposition 6.3. *Let $a \ne 0, c, d \in \mathbb{R}$ and $\alpha \in \mathbb{R}, \rho, \eta < 0$. If $f(x) = ax^\alpha(1 + cx^\rho + dx^{2\rho}(1+o(1))), x \to \infty$, then $f \in 3\mathrm{RV}_{\alpha,\rho,\rho}$ with auxiliary functions A, B given by*

$$A(t) = \frac{\rho c t^\rho}{1 + ct^\rho}, \quad B(t) = \frac{2d}{c}t^\rho.$$

Further for $\alpha \ne 0$ as $t \to f(\infty) := \lim_{x\to\infty} f(x)$

$$f^{\leftarrow}(t) = \left(\frac{t}{a}\right)^{1/\alpha}\left(1 - \frac{c}{\alpha}\left(\frac{t}{a}\right)^{\rho/\alpha} + \left(\frac{c^2}{2\alpha^2}(1+\alpha+2\rho) - \frac{d}{\alpha}\right)\left(\frac{t}{a}\right)^{2\rho/\alpha}(1+o(1))\right).$$

Proof of Proposition 6.3 The first claim follows by the definition of third-order regularly varying functions. We only present the proof of the second claim for $\alpha > 0$ since the case $\alpha < 0$ follows by the similar arguments. Since $f \in 2\mathrm{RV}_{\alpha,\rho}$ with auxiliary function A, we have by Proposition 2.5 in [24] that

$$\begin{aligned}(f^{\leftarrow}(t))^\rho &= \left(\frac{t}{a}\right)^{\rho/\alpha}\left(1 - \frac{c}{\alpha}\left(\frac{t}{a}\right)^{\rho/\alpha}(1+o(1))\right)^\rho\\ &= \left(\frac{t}{a}\right)^{\rho/\alpha}\left(1 - \frac{\rho c}{\alpha}\left(\frac{t}{a}\right)^{\rho/\alpha}(1+o(1))\right), \quad t \to f(\infty).\end{aligned} \tag{6.5}$$

By Theorem 1.5.12 in [6] we have $f(f^{\leftarrow}(t)) \sim t, t \to f(\infty)$. Consequently

$$\begin{aligned}f^{\leftarrow}(t) &= \left(\frac{t}{a}\right)^{1/\alpha}\left(1 + c(f^{\leftarrow}(t))^\rho + d(f^{\leftarrow}(t))^{2\rho}(1+o(1))\right)^{-1/\alpha}\\ &= \left(\frac{t}{a}\right)^{1/\alpha}\left(1 - \frac{c}{\alpha}(f^{\leftarrow}(t))^\rho + \left(\frac{(1+\alpha)c^2}{2\alpha^2} - \frac{d}{\alpha}\right)(f^{\leftarrow}(t))^{2\rho}(1+o(1))\right)\end{aligned}$$

which together with (6.5) implies the desired result. □

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